The Secrets of π

Why the circle can't be squared

Everything is mathematical

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Joaquín Navarro

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This mysterious 3.14159…, which seeps through all doors and windows, which slides down any chimney.

Augustus de Morgan

Preface

The world of numbers is inexhaustible, unending. The bad thing – the only bad thing – is that the deeper we look, the more complicated it becomes. If you want to know more, there is no choice but to get ready for some hard thinking. The study of numbers has led to number theory, now a sturdy branch on the leafy tree of mathematics.

Number theory groups numbers into sets, some obvious, others more counterintuitive. The sets include prime numbers, abundant numbers, transcendental numbers, rational numbers, random numbers, universe numbers, computable numbers, normal numbers, real numbers, hyperreal numbers, transfinite numbers, figurative numbers, complex numbers, pseudoprime numbers, untouchable numbers, apocalyptic numbers, even friendly numbers. As you can see, the list is long and is getting ever longer.

But where does the fascination with numbers come from? Why do so many people dislike the number 13? The number 666 is rather disturbingly known as the Number of the Beast thanks to a mention in Revelation, the last book in the Bible. Why is it called by that name? How do we know that a triangle with sides 2166969314861378833054797972928630716401520276869946534608169199233884599269**6** and 2166969314861378833054797972928630716401520276869946534608169199233884599269**7** long must contain a right angle? Did you know that some brainy researchers have devoted their time to a class of numbers described as "narcissistic"? Perhaps when you consider that they also think about palindromic, tetradic and friendly numbers then there are few surprises left.

There are numbers for all tastes, each with its definition in tow, and π is no exception. It belongs to the transcendental numbers and is suspected to also belong to the normals and a few others. It is also the most studied and admired string of numerals in history, and there is so much information, so many books and websites about π, that to attempt to write something new about it is practically impossible. For this reason, this book is content to go through a rigorous and hopefully entertaining number mania that instills a fascination for π. However, we aim to make the finer points comprehensible and accessible to any interested reader.

Unfortunately – as Euclid explained to Ptolemy I the ruler of Egypt – "There is no quick way to learn geometry," and to know numbers and move between them requires a certain amount of mental effort. So abandon all hope that reading a few pages about mathematics will make it appear easy. Mathematics cannot be fathomed in a hurry, but for that very reason the rewards are all the greater.

Now, how far into this infinite subject should we delve? Why do we calculate the decimal places of π? In reality, what good is knowing the first billion digits? Its digits are endless and they never seem to settle into a pattern, and it's possible that such as pattern does not exist – it is beyond reach with our latest understanding. Is there a limit to the knowledge of π and its digits? Pure mathematics often awakens such questions about its own usefulness. Perhaps the best answer to such a reasonable query is one offered by the famous German mathematician Carl Gustav Jacobi, who defended the subject in 1830 as a means "to honour the human spirit". Let us start this book inspired by that spirit. It is not about knowing everything or even knowing a lot, nor that our wisdom can always be useful. It is about knowing some numerically interesting or funny ideas simply because it is beautiful to know. For the honour of the human spirit.

Chapter 1

Everything you always wanted to know about π but were afraid to ask

In a circumference, the beginning and the end meet.
Heraclitus

π is the the most famous number in the world, the most studied, the most calculated, the most referenced... But nevertheless there is little definite that can be said about π. To begin its decimals begin in this way:

3.1415926535 8979323846 2643383279 5028841971 6939937510...,

and with those fifty magical digits one can navigate around almost the whole wide world of calculations, although it is rare to find a problem in physics or maths that requires knowing more than ten digits of π. In reality, the approximations 3.14 or 3.1416 is perfectly good for basic calculations.

Isaac Asimov once summed this up: "If the universe were spherical and had a diameter of 80,000 million light years, the error in calculating its celestial equator with a value of π to 35 places would be less than a millionth of a centimetre."

If we locate ourselves at a point on Earth's equator, and write the decimal expansion of π with numbers similar in size to those in this book, the current figures calculated by computer would make more than 500 laps around the world. We know that the sequence 0123456789 is found starting at decimal number 17,387,594,880 of π. How small the statement made by the eminent Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966) now seems. He maintained that the search for this specific sequence was a meaningless task because we never would know enough digits of π!

In the 21st century we have found an indisputable use for the decimal expansion of π. When you want to test the performance of a supercomputer you task it with something difficult to calculate, but, at the same time, is well known; the digits of π are the ideal solution.

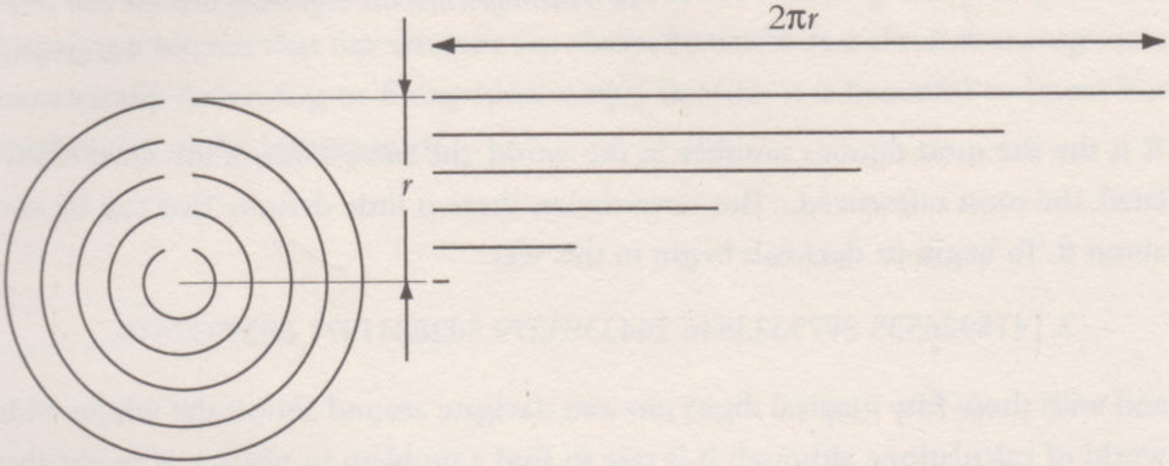
Reinventing the wheel

The number π didn't come from nothing, of course. It was born of simple observations. The ratio or relationship between the circumference of a circle *(C)* and its diameter *(d)* is a constant:

$$C/d = \pi.$$

Or, if preferred,

$$C = \pi d = \pi 2r = 2\pi r.$$



The ratio of the circumference to diameter is a constant, a fact discovered by intuition and simple observation. When the diameter grows (the diameter is twice the radius r), it grows in proportion to the length of the circumference (perimeter).

Simple observation tells us that this relationship is a fixed number, because the greater the diameter of a wheel, the greater (proportionally) the distance travelled by a fixed point on it with every revolution. Thus:

Length of a circumference / diameter of the circle = constant ≈ 3.14.

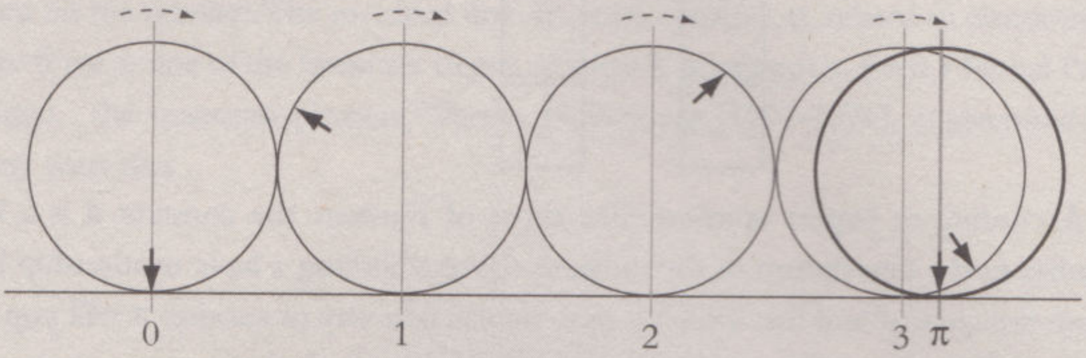
Or, in algebraic terms, where l is the distance around the circle and r its radius (diameter, d, double r):

$$l = \pi d = \pi 2r = 2\pi r \approx 2 \cdot 3{,}14\ r.$$

The sign ≈ means 'approximately equal', and most of the history of π concerns reducing the approximation to the smallest margin by adding more and more digits after 3.14...

THE FIRST RECTIFICATION

Measuring the length of a curved line is called rectification. Perhaps the most obvious thing to rectify is the circumference of a circle.

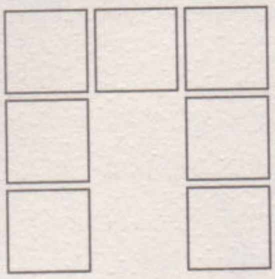


When a circle with perimeter *p* rolls without slipping along a line, it makes one complete turn by travelling π times the diameter *d*. This process, translating the perimeter into a line, is called rectification. Rectification tells us that $p / d = \pi$.

Mathematicians have spent much of their ingenuity in the calculation of π as accurately as possible, honing every decimal place with valiant effort. For a time it was thought these feats would eventually come to an end, but in 1862 the German mathematician Ferdinandvon Lindemann (1852-1939) ended the long search with a definitive answer – of sorts: The number π is an infinitely long string of digits. There is not and never will be a way to find the 'exact' value of π. Throughout these pages we will try to explain why.

Initially, the number π was not known as such. Although mathematicians like William Oughtred (1574-1660), Isaac Barrow (1630-1677) and David Gregory (1659-1708) used this spelling, the term did not become 'official' until William Jones (1675-1749) referred to the constant in this way in 1706, in one of his *Synopsis Palmariorium Mathesios*. π is the Greek letter 'p', and is the initial of the word 'periphery', which in Greek is written περιφρεια. Later, the great Leonhard Euler (1707-1783), after initially using a 'c' and a 'p' also began to use the symbol π, and slowly but surely the idea spread. However, in 20th century Egypt the constant was denoted with the Arabic letter *ta* instead of *pi* largely for nationalistic reasons.

Today π is used in mathematics especially to symbolise the number we have indicated, but it is not the only use of the letter. For example, $\pi(x)$ is used to indicate the function of natural numbers 'number of primes smaller than x'. And going down to trivial mathematical pursuits, π is the name of the heptomino, a figure of seven connected squares, as depicted here:



According to expert opinions, like those of Einstein, the constant π is a key number in the description of the universe. By establishing a basic relationship between the circular and that which is not circular, one way or another π will pop up like a cork on the surface of water throughout all natural phenomenon governed

THE LITTLE PRINCE'S ASTEROID

Every circle contains a curiosity, a simple one but no less surprising nonetheless. It is true of any circle that

$$\text{length} / \text{diameter} = \text{constant}$$

In other words a multiplier in the denominator of this fraction determines the same multiplier for the numerator. We will illustrate this with a simple example. In *The Little Prince*, a story by French writer and aviator Antoine de Saint-Exupéry (1900-1944), the protagonist goes around his small asteroid sweeping volcanoes. We will assume that he travels along a meridian, because otherwise the story would not work. The prince is exactly 1 m tall. If he walks 1,000 m, what distance does his head travel? Let's start by remembering that our unit of measure is metres. As the prince walks 1,000 m and the

$$\text{length of the circumference} = 2\pi r,$$

it is clear that the

$$\text{distance in metres travelled by foot} = 1{,}000 = 2\pi r.$$

As the protagonist is 1 m tall, letting C be the distance travelled by the head, we have

$$C = 2\pi (r + 1).$$

And, by subtracting the first expression from the second, we obtain:

by laws that have something to do with the curved shapes or rotation. Along with a handful of other constants, we will never part company with π.

Approaching the issue from another direction, there are many people, mostly numerology enthusiasts, who see π everywhere as if there were a conspiracy theory based on the number. The so-called fine structure constant, α, related to electromagnetic force, is one of the favourite targets of these π worshippers. Even a Nobel Prize winner, the quantum physicist Werner Heisenberg (1901-1976), conjectured for many years that

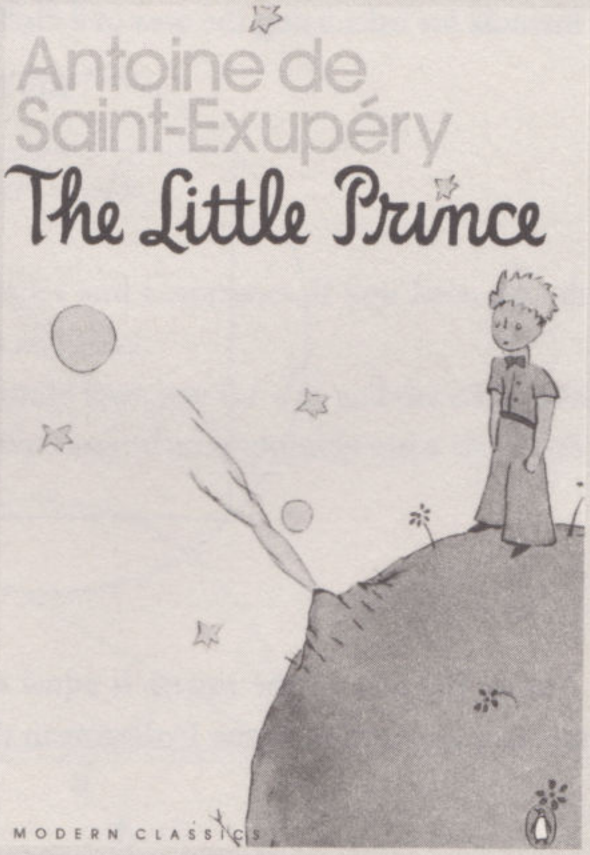
$$1/\alpha = 2^4 3^3/\pi.$$

---

distance in metres travelled by the head - distance in metres travelled by foot =

$$= C - 1.000 = 2\pi\,(r+1) - 2\pi r = 2\pi\,(r+1-r) = 2\pi \approx 6{,}28.$$

The difference is 6.28 m, but the curious thing about the whole calculation is that the radius of the asteroid has no influence on it. In fact, whatever the measure of the original radius, any addition of 1 m to the original radius only increases the length of the circumference by 6.28 m. If the asteroid's radius had measured 1,000 km, when the prince swept along 1,000 km the additional distance travelled by his head with repsect to his feet would be exactly the same: 6.28 m.



Cover of The Little Prince *by Antoine de Saint-Exupéry.*

However, Heisenberg was not alone. Throughout the theories of physics there are similar approximations, such as

$$\frac{1}{\alpha}=\left(\frac{8\pi^4}{9}\right)\left(\frac{2^4 5!}{\pi^5}\right)^{\frac{1}{4}},$$

$$\frac{1}{\alpha}=108\pi\left(\frac{8}{1843}\right)^{\frac{1}{6}},$$

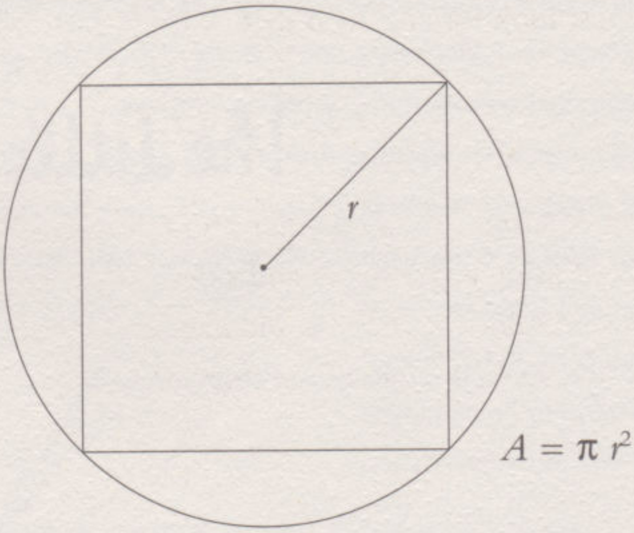
that involve π and have a lot of merit.

The centuries-old problem

The number π is not only the constant ratio between the length of the circumference to its diameter, but is twice the constant ratio between the area of a circle and its inscribed square. We have all be taught at school,

$$A=\pi r^2,$$

the formula for calculating the area of a circle of radius r.



And as the area of the square is equal to precisely the square of the side, an elementary application of the Pythagorean theorem leads to

$$\frac{A}{\textit{Area of the inscribed square}}=\frac{\pi r^2}{2r^2}=\frac{\pi}{2}.$$

But who assures us that this constant, this π derived from the area, is the same π that is linked to the length? It appears to be the same π, but contrary to what we learn in school it is not completely obvious that the two constants are the same. We had to wait for the genius of Archimedes to be sure.

The problem that really mattered to the ancient mathematicians was to build a square area equivalent to a circular area. And this was very much a practical problem, since measuring a square area was a simple task, while measuring circular areas presented some difficulties and offered only an approximate value. Greek thinkers believed that they could find some kind of divine procedure, clean and quick, much along the lines of their philosophy, that could construct a square equivalent to a circle using only a ruler and compasses. That is precisely what is meant by 'squaring the circle', to obtain a shape that meets this equivalence using ruler and compasses in a finite number of steps. Mathematics advanced in this direction, always after this elusive quadrature carrot but without ever biting it.

Throughout the centuries, all geometers sought to square the circle, which is to construct π exactly with ruler and compasses, or to get just a little closer by adding another exact decimal to the number π. Algebraically, 'squaring' a circle of area πr^2 means searching for a square of side l, so that

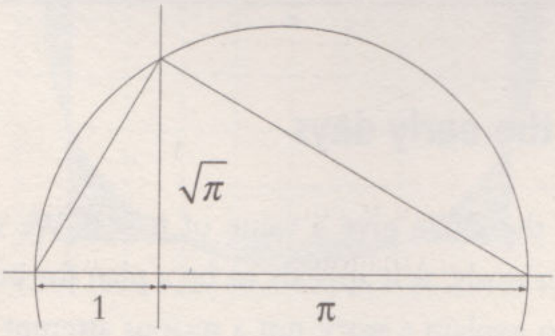
$$\pi r^2 = l^2.$$

That is, one must look for an l such that

$$l = \sqrt{\pi r^2} = r\sqrt{\pi},$$

which means constructing $\sqrt{\pi}$ with ruler and compasses. If you have $\sqrt{\pi}$, then you can also make π using a compasses and ruler.

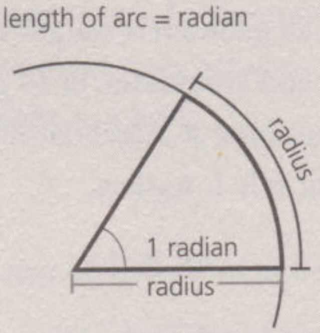
If we had a precise value for π, we would have one for $\sqrt{\pi}$, and the circle would be squared. What follows is, in essence, the story of an impossible quest that takes us



increasingly forward but never closer to the goal. One geometer at a time (no doub people of great ingenuity) added a decimal to π, and increasingly, indirectly, all other mathematicians advanced a step forward.

THE RADIAN AND π

In mathematics, angles are not measured in sexagesimal (base 60) degrees, minutes and seconds, not even in sophisticated gradians (one four-hundedth of a tull 360° turn). The advent of calculus (differentials, integrals, etc.) led to the use of a more natural measure, although at first it seems rather complicated. The radian is defined as the angle that determines an arc along the circumference of equal length to the radius.



As the length of its perimeter is equal to $2\pi r$, the entire circumference is an arc with an angle of 2π radians. So that

$$1 \text{ radian} = 360/2\pi \text{ sexagesimal degrees} \approx 57°17'45''.$$

The most common equivalences are:

$$30° = \pi/6;\ 60° = \pi/3;\ 90° = \pi/2;\ 180° = \pi;\ 360° = 2\pi.$$

The history of π: the early days

A couple of verses of the Bible give a value of $\pi = 3$. We will not take this approximate value too seriously, as it appears to be a plan for building a circular altar and is just an excuse to explain a story, not a serious attempt to calculate π. However, for the curious reader, what the Bible says, for example in *Kings* (I, 7, 23) is:

"And he made a molten sea, ten cubits from the one brim to the other: It was round all about, and his height was five cubits: and a line of thirty cubits did compasses it round about."

The calculation of the value of π used here, which ends with $\pi = 3$, we leave in the hands of the astute reader.

The Egyptian Rhind Papyrus, one of the world's oldest mathematical texts, which experts place around 1650 BC, is also an indirect reference to π. Problem number 50 (of the 87 listed), says: "A circular field has diameter of 9 khet (1 khet ≈ 50 m). What is its area?" In modern terms, this amounts to an area equal to

$$\pi\left(\frac{9}{2}\right)^2 = \pi\frac{81}{4}.$$

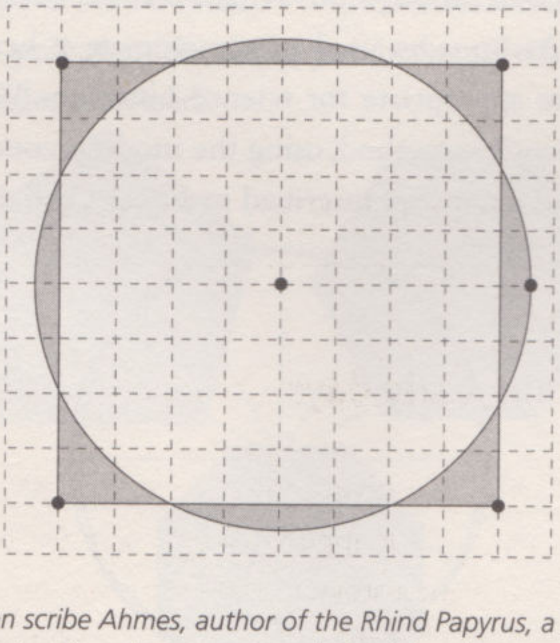
But the papyrus itself provides the system to calculate the area, which is

$$\frac{64}{81}d^2,$$

where d is the diameter. As $d = 9$, we have that

$$\pi\frac{81}{4} = \frac{64}{81}d^2 = \frac{64}{81}9^2 = \frac{64}{81}81;$$

$$\pi = \frac{256}{81} \approx 3.160493827.$$



According to the Egyptian scribe Ahmes, author of the Rhind Papyrus, a square with a side of 8 is equivalent in surface area to a circle with a diameter of 9.

However, this approach is inferior to that presumably obtained by the Egyptians in Giza, back in 2600 BC. The perimeter/height ratio of this city's pyramids is 22/7, but it ignores the reason for this number. We often suppose that the ancient architects considered it to have some divine quality. The truth is that many take it as an approximation to π, perhaps seen as a mystical figure by the builders. If we accept this as an explanation and we assume that the perimeter/height ratio is not a mere coincidence, then we get

$$\pi = 22/7 = 3.142...,$$

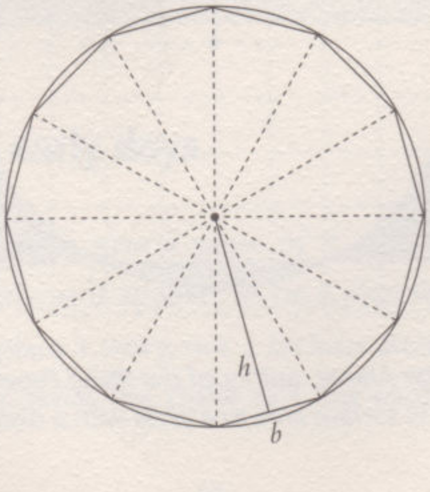
a figure that is not all that bad.

In Babylon, progress took place more slowly. On a tablet in the ancient city of Susa, dating back to 2000 BC, the value of π was $25/8 = 3.125$.

In India, Vedic texts from 9th century BC provide different values of π based on practical issues. The best value comes from astronomical calculations and is found in the *Shatapatha Brahmana*; $\pi = 339/108 = 3.1388...$

The history of π: Archimedes

We now arrive in ancient Greek era, in which one of the greatest minds of humanity, that of Archimedes of Syracuse, was at work. It is possible that Anaxagoras dealt with π in 5th century BC, although no written records were kept. We will not reproduce the method Archimedes used to approximate π here, as it is a long and complex process, more appropriate for science historians. We will try to explain with a more comprehensible method, using the modern concept of limits (*lim*). To begin with, we suppose a polygon inscribed in a circle, like that in the illustration.



We can see that several triangles are formed of base *b* and height *h*. The total area of the *n* triangles is close to but less than the area of the circle around them:

$$A_n = n(\text{ area of triangle }).$$

Such that

$$A_n = n\left(\frac{1}{2}hb\right) = \frac{1}{2}h(nb).$$

To move toward equality, more and more triangles are added so $n \to \infty$,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\frac{1}{2}h(nb) = \frac{1}{2}\lim_{n\to\infty} hnb = \frac{1}{2}\lim_{n\to\infty} h \cdot \lim_{n\to\infty} nb = \frac{1}{2}r \cdot 2\pi r = \pi r^2,$$

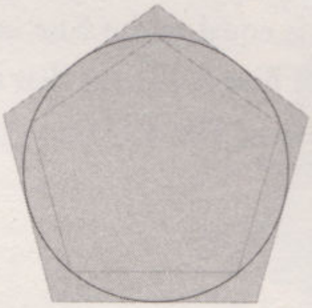
as

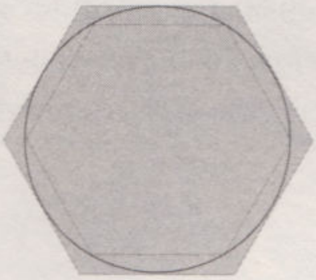
$$\lim_{n\to\infty} h = r,$$

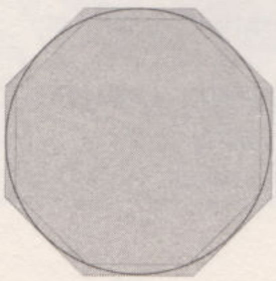
and we reach the following conclusion:

$$\lim_{n\to\infty} nb = \text{ length of circumference } = 2\pi r.$$

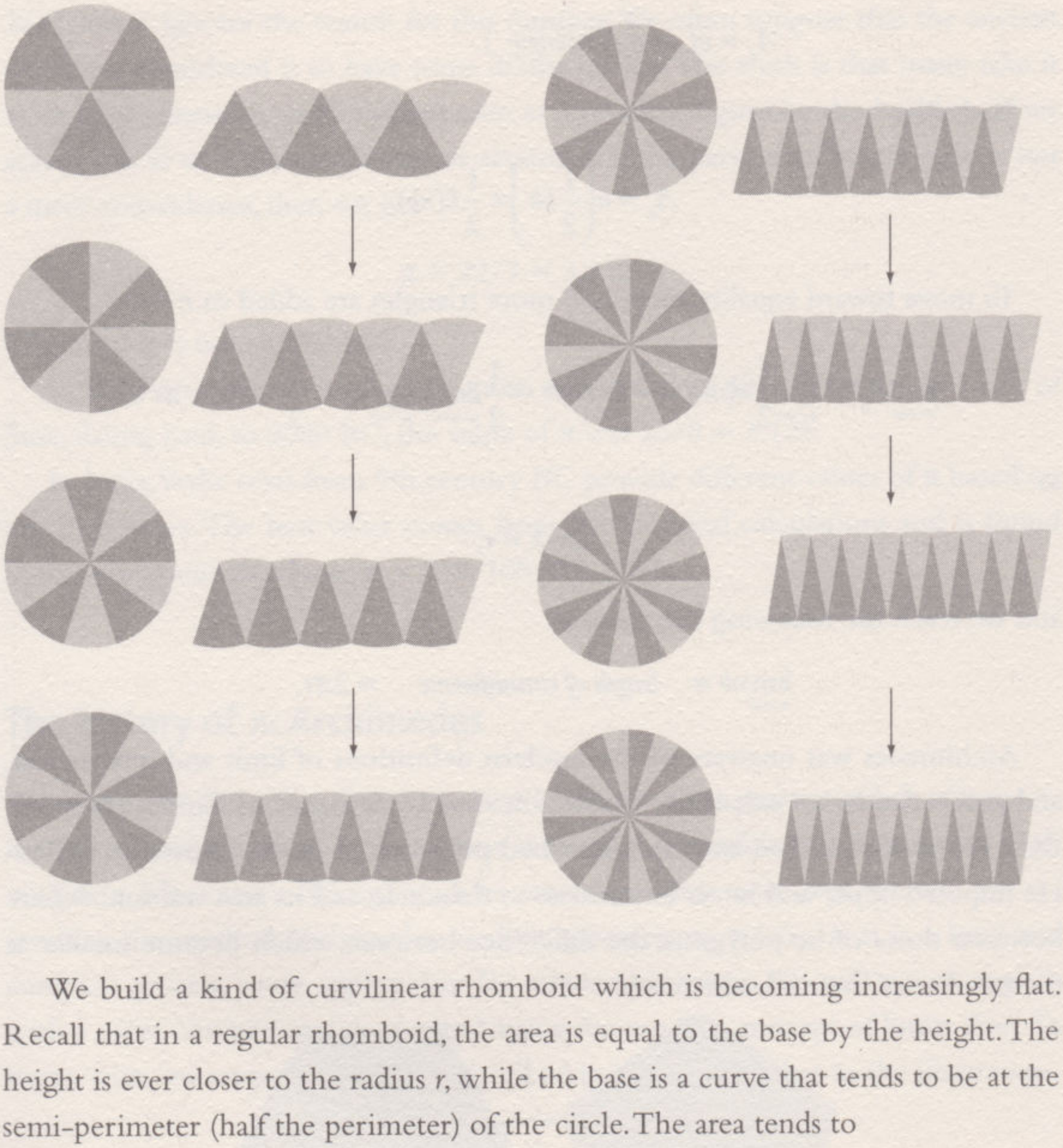
Archimedes was unaware of the modern definitions of limit and integration, and proceeded by an exhaustive method devised by Eudoxus of Cnidus (400-347 BC). He used inscribed and circumscribed polygons, as in the illustration below. He imposed upper and lower boundaries to the circle, and its area was somewhere between those of he polygons, the difference between which became smaller as the number of their sides increased.







An outline of what such a 'voyage to the limit' would be like today may help to understand why the area of the circle is πr^2:



We build a kind of curvilinear rhomboid which is becoming increasingly flat. Recall that in a regular rhomboid, the area is equal to the base by the height. The height is ever closer to the radius r, while the base is a curve that tends to be at the semi-perimeter (half the perimeter) of the circle. The area tends to

$$r\frac{l}{2}=r\frac{2\pi r}{2}=r\pi r=\pi r^{2}.$$

Archimedes came to a fork in the value of π,

$$223/71 = 3.140845\ldots < \pi < 22/7 = 3.142857\ldots,$$

which was an incredibly accurate result.

ARCHIMEDES OF SYRACUSE (c. 287 BC – C. 212 BC)

Engineer, physicist, astronomer and Greek mathematician, Archimedes is considered the greatest scientist of ancient times and one of the greatest minds in history. His achievements earn him a position matched only by the likes of Newton, Gauss or Von Neumann. His contributions to other sciences are legion too. As an engineer and physicist he created the screw pump that takes his name, parabolic mirrors, and numerous applications of lever systems and pulleys, among many other inventions. Perhaps his more universal contribution is the one he made to hydrostatics, known as Archimedes' principle or the principle of buoyancy. The image of Archimedes jumping out naked from the bath and shouting *"Eureka!"* ("I've found it!") is now an iconic momement in the history of knowledge.

As a mathematician his findings are almost countless: apart from the approximation of π he dealt with finding the perimeter, area, volume or centre of gravity of many geometric shapes (spheres, cylinders, parabolas, spirals...), he studied Diophantine equations, he helped conceive and count large numbers, and so on.

He died during the siege of Syracuse when he and his machines helped defend against the Romans, albeit in vain. According to the account from Plutarch, Archimedes shouted at an invading soldier who was about to burn his drawings. The soldier did not like this and killed the great master with his sword. Plutarch states that the death of Archimedes shocked the Roman commander, who considered him a very valuable booty. On his tomb there is a sphere inscribed in a cylinder, giving the ratios of their volumes and areas, a discovery that Archimedes himself made.

One of the inventions attributed to Archimedes is the use of mirrors that reflected sunlight so that it fell on the Roman ships and burned them. The aim was to protect the city of Syracuse from the Roman naval siege.

The method used by Archimedes would become the default technique for centuries. It is natural, accessible and direct; a marvel of geometric ingenuity. In essence, Archimedes devised an algorithm, instructions for calculating π as accurately as you needed. This algorithm can be applied with a calculator, a computer or a recurring calculation. If n is the side of a polygon circumscribed or inscribed in a circle, and a_n and b_n their perimeters

$$a_{2n} = \frac{2a_n b_n}{a_n + b_n},$$
$$b_{2n} = \sqrt{a_{2n} b_n},$$

we obtain the so-called Archimedean algorithm, a recursive formula that gets ever closer to π as n grows. And in any case,

$$a_k > \pi > b_k.$$

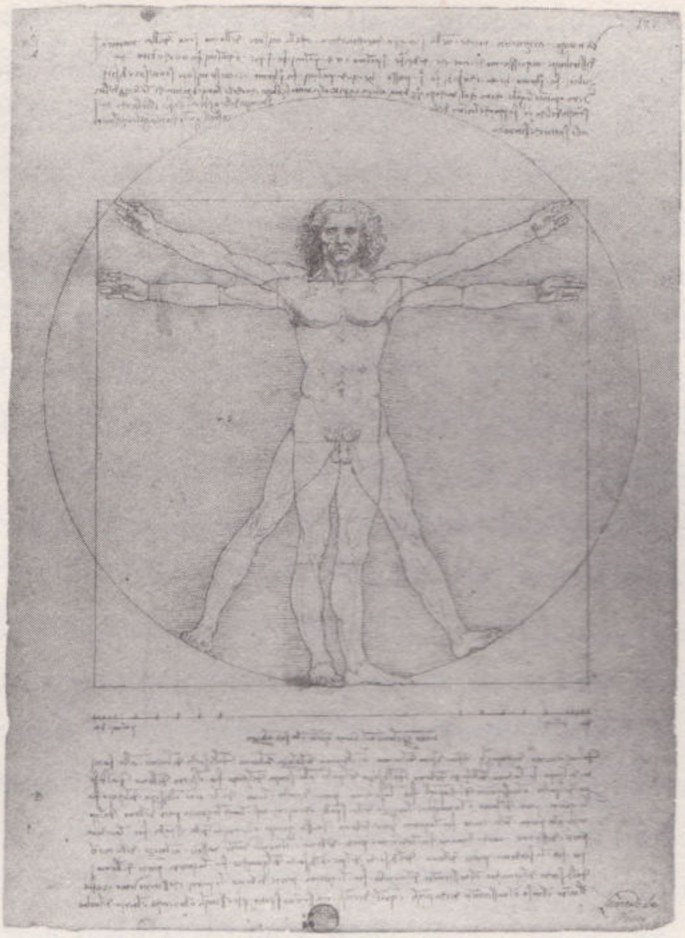
The Archimedean algorithm, starting from a regular hexagon, with $a_0 = 4\sqrt{3}$ and $b_0 = 6$, provides in sequential order:

$$3.00000 < \pi < 3.46410$$
$$3.10583 < \pi < 3.21539$$
$$3.13263 < \pi < 3.15966$$
$$3.13935 < \pi < 3.14609$$
$$3.14103 < \pi < 3.14271$$

The approximation presented by Archimedes is the inequality from a polygon with 96 sides expressed in rational numbers.

The history of π: from Archimedes onwards

By 20 BC, the famous Roman architect, military engineer and writer Marcus Pollio Vitruvius (85 BC – 20 AD), better known in our time as Vitruvius, wrote the treatise *De Architectura* (a total of 10 volumes), in which he used the Mesopotamian value of π = 25/8. Vitruvius tackled the value of π again using the empirical measure of a marked wheel. However, he did not earn his fame from this calculation. He is best known from a drawing by Leonardo da Vinci known as the Vitruvian Man, which



The Vitruvian Man or the Proportions of the Human Body According to the Roman Architect that Put Them Forth*; the famous drawing by Leonardo da Vinci (1452-1519).*

measures the human proportion. Despite his fame, Vitruvius did not improve upon the approximation of Archimedes. However, Ptolemy (c. 100–170 AD), a Hellenised Egyptian (hailing from Greece), who was an astronomer, astrologer and geographer, did with his fraction $\pi = 377/120 = 3.141666\ldots$ To get this value he used a 120-sided polygon, concluding that $\pi = 3 + 17/120$, all in all a magnificent result. In any case, posterity did not grant him his amply deserved tribute, though it did for another of his works, *The Almagest,* thirteen books the title of which is translated as "The Great Treaty". This work attempted an explanation of everything known – including the now infamous geocentric model of the Universe which was accepted as fact until the arrival of Copernicus 1,300 years later.

The attention given to the developments in the Western world often detracts from the achievements of other ancient cultures, apart from the Babylonians, Greeks, Romans and Egyptians. While the West was shining a light on π, what happened in the Far East?

In China, for example, leaving aside the tentative attempts of Chan T'sang (around 220 BC), who established the value of π at 3, and the work of a few other mathematicians, Zhang Heng (78-139 AD) found time among his many activities (he was dedicated to astronomy as well as mathematics and even devised an earthquake detector) to record an approximation of $\pi = 736/232 = 3.1724\ldots$ in one of his books. While calculating the volume of a sphere inscribed in a cube he made use of the approximation $\pi = \sqrt{10} = 3.162277\ldots$

The name Wang Fang or Fan (217-257 AD) is known today for the malformation that bears his name – people born with their feet facing backwards are said to to have Wang Fang feet. However, Wang Fang, whose feet were normal, was also interested in mathematics, and calculated some figures of π. In the year 250 he gave the approximate fraction $\pi = 142/45 = 3.155555\ldots$

The mathematician Liu Hui (c. 220–280 AD) wrote some comments in *Nine Chapters on the Art of Mathematics* in the year 263, thanks to which we now know something of his life and achievements. In them we find a recurrent formula that calculates the sides of a regular polygon with 3×2^k sides, if those of a polygon with $3 \times 2^{k-1}$ sides are already known; Liu Hui recommended using the approximation $\pi \approx 3.14$, though he went further and came to $\pi = 3.141592104\ldots$; this suggest that that he worked with a polygon with 3,072 sides.

A REFIND VALUE OF π 3RD CENTURY AD
圓周率
LUI HUI'S NINE CHAPTER ON THE MATHEMATICS AD 264
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Stamp in 1999 in the Federated States of Micronesia showing the method of calculating π developed by the Chinese mathematician Liu Hui.

Centuries later, Zu Chongzhi (429-500 AD), a scientist and mathematician who devised calendars, built π in an admirable way:

$$3.1415926 < \pi < 3.1415927$$

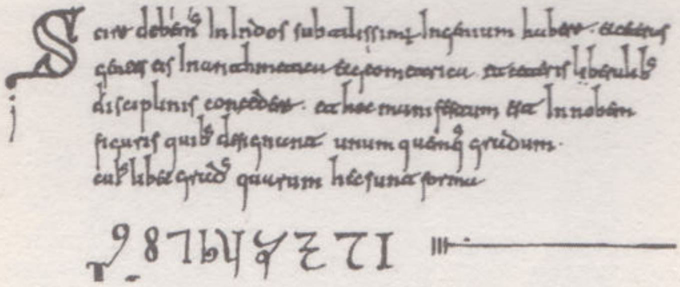
He even recommended using fractions such as 22/7 for simple applications and 355/113 for the most complicated.

Let us leave China for a moment to go to India, where Aryabhata (c. 476-550 AD), the great patriarch of the Indian sages, obtained a value of π of 3.1416 with the help of a 384-sided polygon.

Brahmagupta (598-665 AD), arguably the most gifted of the Indian mathematicians, is the author of a long text, the *Brahmasphutasiddhanta*, where, not without some disappointment, we again find the value $\pi = \sqrt{10} = 3.162277...$

We must move to the 12th century to find something better at the hands of Bhaskara II 'The Teacher' (1114-1185), recorded in his book *Lilavati*. The book is named after his daughter, who, judging by the merit of the work alone must have been a beautiful young woman, because her name means 'beautiful'. Bhaskara gives the approximation of $\pi \approx 3917/1250 = 3.1416$.

Our numbering system, which is positional and in base 10, has the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, is of Indo-Arabic origin. We do not usually pay much attention to this fact but it is the foundation of Western commerce. The introduction of such instruments to the competitive culture of the West would bring arithmetic into everyone's reach.



Indo-Arabic digits, as they first appear in Spain (974-976) in the Codex Albeldensis, *also known as the* Codex Vigilanus *after the monk that compiled it.*

This book is not best suited to narrate the vicissitudes of the Indo-Arabic numbers, but we know that is where our digits came from. Surprisingly these numerals and their number system were not introduced to Europe until the 10th century. They were presented in the *Liber Abaci* by Leonardo of Pisa (c. 1170-1250), better known as Fibonacci. The popularity of the numbers spread rapidly, especially among merchants. The calculations in the new system were simple compared to the nightmare, of Roman numbering. They were simplified further thanks to algorithms of multiplication and division and, albeit slowly, civilisation took a step forward.

In the middle of our history is Fibonacci, who in 1220 gave the approximate value to π of 3.141818 in one of his works, *Practica Geometricae*, using the same method as Archimedes, albeit a little freely.

But let's not get ahead of ourselves. Long before this we have to stop in the Muslim world at the door of the leading figure of Abū 'Abdallāh Muhammad ibn Mūsā al-Khwārizmī (c. 780-850 AD), whose name also appears as al-Juarismi and similar spellings. In any case, the name of the Persian mathematician is the origin of the words *algorithm*; he also penned a work entitled *The Compendious Book on Calculation by Completion and Balancing*, which resulted in nothing less than the invention of algebra (another word borrowed from Arabic). When his works reached the West, they had an extraordinary influence. Al-Khwārizmī generally recommended the use of 3.14 for easy calculations and 3.1416 for large-scale astronomical ones.

A likeness of al-Khwārizmī with heroic features adorns a Soviet stamp of 1983. The Soviets saw him as an Uzbek, since he was born in what is now Uzbekistan – one of the Soviet republics when the stamp was released.

In 1424, another Persian, Jamshīd al-Kāshī (1380-1429), who was based in the court of Tamburlaine in Samarkand, did not calculate π, but 2π, and did so using sexagesimal numbering. This counts in base 60 so we would write $1/60 = 0.1$, $1/60^2 = 1/360 = 0.01$, etc. He arrived at 9 figures, which translated to decimal provides an approximation of π correct to 16 digits. Al-Kāshī computed:

$$2\pi = 6 + \frac{16}{60} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{1}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9} + \dots$$

using polygons with 3×2^{28} sides. This value exceeded the mark of thirteen decimals obtained by Madhava of Sangamagrama (c. about 1350-1425) in India a few years earlier, in 1400. Nevertheless, Madhava's estimates had an original feature. A series was used for the first time as a way of representing an infinitely long number, using pure maths to evaluate π. The Madhava formula is none other than the one that has passed down through Western history with the title of the 'Leibniz formula', only Madhava had discovered it much earlier:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

The convergence of this series is very weak, and you have to add and subtract thousands of terms to get decent results. Madhava transformed it into

$$\pi = \sqrt{12}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \dots\right),$$

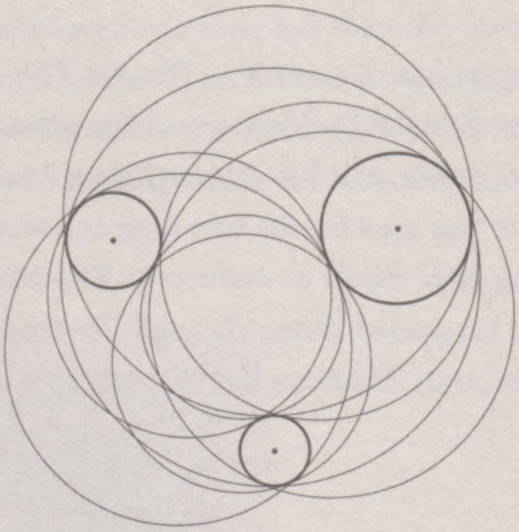
which he used to calculate π.

In 1573, the German Valentinus Othoor (Valentin Otto; c. 1550-1603), a fan of Copernicus, recommended using $\pi = 355/113 \approx 3.1415929\ldots$, but this is nothing compared with the results obtained later, and no thanks to this particular approximation despite the intelligence displayed in achieving it. The ninth number of π reached by the Frenchman François Viète is nothing extraordinary. To get that result he used Archimedean methods and worked with a 393,216 ($= 6 \times 2^{16}$) sided shape. The important thing here that is the formula he created for π was too difficult to use because it involves extracting square root after square root. In modern notation, Viète's work would be written as

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} \cdot \dots$$

The path followed to achieve this is explained, along with other formulae, in Chapter 4.

FRANÇOIS VIÈTE (1540-1603)



The Problem of Apollonius takes three circles and tries to find any circles that are tangential to all of them. Tradition dictated that the solution could be drawn using only ruler and compass. In the general case, there are eight different solutions.

You could not say that Vieta (the Latinised form of his name used in Europe) was a professional mathematician. He was primarily a lawyer and, upon Henry IV's ascension to the French throne he reached the position of courtesan and a member of the king's inner circle. His fame, somewhat legendary, is that he was a pioneer in cryptography, as he would decipher encrypted messages sent by Philip II of Spain, his master's enemy. The Spanish monarch came to think that the evil French king held some kind of pact with the devil, because he seemed to know about diplomatic moves in advance – and all this with Philip being sure that he had God on his side. Vieta was also a very good geometer, but an almost better algebraist, delving into the fields of trigonometry and equation solving and, perhaps more importantly, he invented modern algebraic concepts that revolutionised the oral and written traditions and promoted the progress of science. He had a great rivalry, later bartered into friendship, with Adriaan van Roomen (1561-1615), who proposed the problem of tangent circles or the Problem of Apollonius.

Viète's rival and friend, the Dutch geometer Adriaan van Roomen (1561-1615), also known as Adrianus Romanus in a culture in which almost all the eminent names were Latinised, devoted more effort than Viète to the study of Archimedes polygonal method, using an impressive 2^{30}-sided figure in 1593 to compute π to 16 exact figures.

But if the work of Van Roomen seems formidable, what about Ludolph van Ceulen (1540-1610)? This German, a true geek for π, first calculated in 1596 up to 20 digits and then to 35:

$$\pi = 3.14159265358979323846264338327950288...$$

Van Ceulen reached such popularity that in many countries π came to be known as Ludolph's number. Such was the affection that Van Ceulen had for pi that his favourite number was engraved on his tomb in the city of Leiden. The direct testimony of such an all-encompassing passion was eventually lost when the tomb was destroyed in World War II. As can be seen in Chapter 5, the edifice was rebuilt in 2000 with glowing digits; the perseverance of Van Ceulen deserved some kind of recognition.

Willebrord Snel van Royen (1580-1626), also called Snell or Snellius, a student of Van Ceulen, is best known in the West as the discoverer of the laws of refraction. As a mathematician he also tackled the computation of π, with 35 correct digits, which were published in 1621 in *Cyclometricus*. The method used to achieve the result is a marked improvement on that of Archimedes. Refinements of Snell's calculation were later made by the great Christiaan Huygens (1629-1695).

In 1630, Christoph Grienberger (1561-1636), a Jesuit Austrian generally known as an astronomer, broke the record at the time for digits of π, reaching 39. Posterity has remembered him by naming a lunar crater after him. An apt memorial for a hard-working astronomer.

With them came scandal… and analysis

The names Gottfried Leibniz and Isaac Newton have achieved immortality in the field of science because they are the creators of infinitesimal analysis, or calculus, the bane of many a mathematics student's early career. Leibniz and Newton led mathematicians to a wonderful world full of new possibilities by taming infinity, and better yet, taught them how to move from the finite to the infinite and return laden with meaningful results. Many greats, such as the visionary Archimedes, roamed this way; Leibniz and Newton walked up behind them, climbed on to their shoulders and pointed out how to enter and exit the maze of the unknown.

Series of powers and integrals are the immediate result of applying mathematical analysis techniques. Calculating π is no longer a simple matter of measuring polygons, but now it is a mathematical question, which involves mainly "the little grey cells" as Hercules Poirot would say.

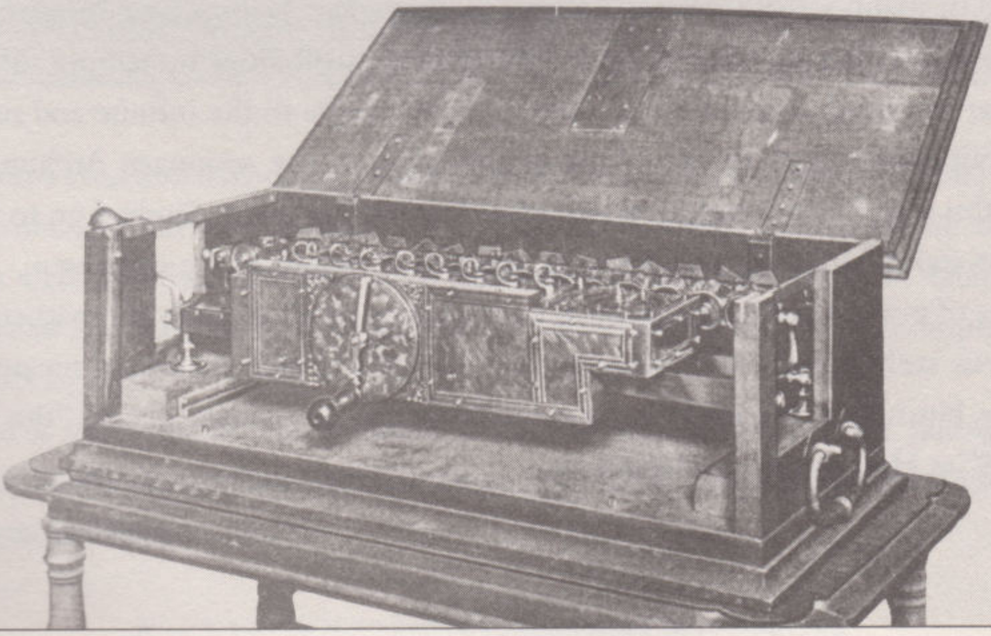
GOTTFRIED WILHELM LEIBNIZ (1646-1716)

To summarise in such a limited space the most important traits of a personality as diverse as that of Leibniz is not an easy task. For instance, his complete published works fill 25 volumes, an estimated total of 200,000 pages. This great thinker was born in Leipzig and dedicated his activities to fields as diverse as law, diplomacy, mathematical logic, religion, historiography, orientalism, binary arithmetic, ethics, physics, biology, engineering and perhaps his most important contribution, infinitesimal and integral calculus.

Leibniz was a child prodigy, he read widely and tackled, apparently with ease, case law and diplomacy. He created the first scientific journal in history, the *Acta Eruditorum*, which published some, but not all, of his discoveries and investigations.

He was always gifted with a sixth sense for the notation of things, and to him we owe, among others, signs as widespread as the integral ($\int$) or the differential (*dx*), and expressions such as "life force". Part of his life was involved in a nasty dispute with the supporters of Newton (with the great Newton behind them) over which of them had discovered calculus. Today we believe that they both did it independently at much the same time by coincidence. Of the mathematician Leibniz we must also recall that he made important contributions to mathematical logic, automata, the binary numbering system and topology, which he called *analysis situs*.

In 1673, Leibniz designed a mechanical calculator capable of performing the four basic arithmetic operations. He built a functional version of it the following year.



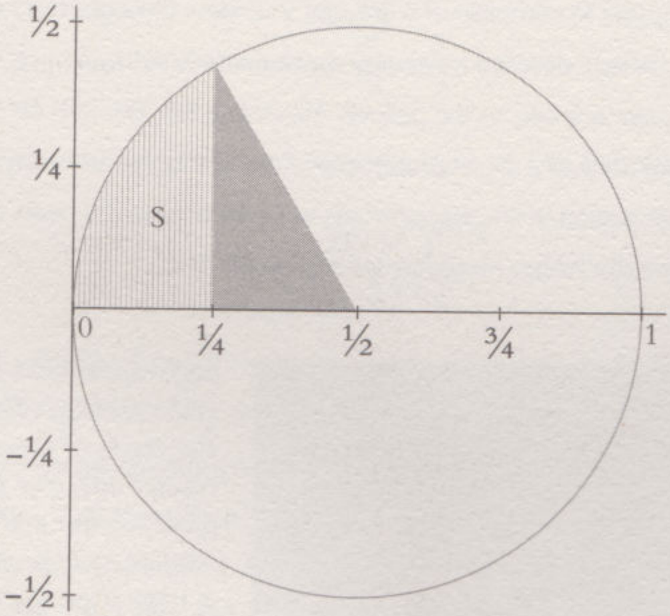
From now on we will ignore the approximations of π coming from the Orient, unless justified by the large number of digits reached or the development of a very innovative procedure.

Newton and Leibniz kept alive an unedifying controversy as to the origin of the development of calculus, and one could say that with them came scandal. We will set out the facts of the dispute in order to focus on the results.

By 1665, the year before the great fire of London, Sir Isaac's brain was apparently unoccupied, as a year later he apologised for having devoted some time to the computation of π only because "he had nothing else to do at the time." Excuses aside, Newton left the binomial formula and discovered the following series

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \dots\right),$$

with which he calculated 16 exact digits of π. As was the case in many other areas of his work, Isaac did not regard this as very significant, and it was not included in any book; the result was only published posthumously. It is always interesting to follow the footsteps of a genius, so we will continue our walk with Newton.



The area of the sector marked on the image is π/24, dividing the circle's area into 6; if we subtract the triangular area, which has a value of $\sqrt{3}/32$, we are left with an area of the circle we have labelled with an S; the equation of the circle in its current position is

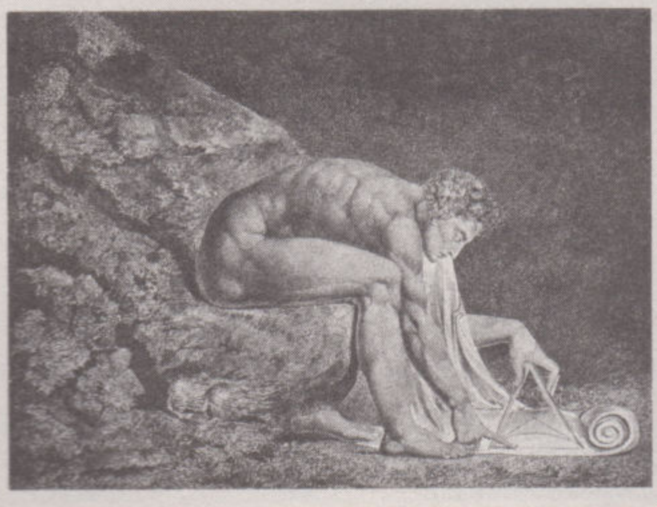
$$y^2 + x^2 = x,$$

SIR ISAAC NEWTON (1642-1727)

Although he is known best as a physicist and mathematician, Newton was a multifaceted character: alchemist, theologian, politician, astronomer, and probably the occasional student of other disciplines. In any case, he is one of the most important scientists in history.

His major work, *Philosophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy),* was published in 1687 mainly because of pressure from his relatives, which proves to be an insight into the the somewhat elusive profile of an uncommunicative Newton. His two best-known scientific contributions, the law of universal gravitation (the legend has it that the discovery was made when an apple fell in his garden) and infinitesimal calculus, contained in the same book. Among his contributions to physics are color theory and diffraction, the first functional reflecting telescope and the corpuscular theory of light. He also formulated the laws of conservation of linear momentum and angular momentum of force. In astronomy, he made brilliant studies of planetary motion and the nature of the orbits. In pure mathematics, apart from the introduction of differential and integral calculus that dominates everything, he also contributed with a lot of power series, the binomial theorem, the propagation of error and the approximation of the zero of the functions.

In the last years of his life, Newton, now an almost silent member of Parliament (his most memorable comment was to complain of a draught that blew through the chamber and ask that the window be closed), devoted his energy as warden of the Royal Mint, which enabled him to send several counterfeiters to the gallows. He always thought that his alchemical and theological discoveries (he was a secret proponent of the Monophysite heresy) would outlive him. Although almost deified in life, Newton was not well liked by his peers and once dead and buried in Westminster Abbey his status quickly diminished.



Isaac Newton in a painting by William Blake (1757-1827). The verses of Alexander Pope (1688-1744) give an idea of the great scientist and his influence: "Nature and Nature's laws lay hid in night: God said 'Let Newton be!' and all was light."

which can be completed in this form $y^2 = x(1-x)$ or $y = \sqrt{x(1-x)} = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$. Using integral calculus invented by Newton himself we get:

$$S = \int_0^{\frac{1}{4}} x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx = \int_0^{\frac{1}{4}} x^{\frac{1}{2}}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \ldots\right)dx =$$

$$= \int_0^{\frac{1}{4}}\left(x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} - \frac{x^{\frac{5}{2}}}{8} - \frac{x^{\frac{7}{2}}}{16} - \frac{5x^{\frac{9}{2}}}{128} - \ldots\right)dx.$$

From there, it's just a matter of integrating term by term and having some skill with numbers... or to be Newton.

His compatriot Abraham Sharp (1651-1742) made use of the following equation of the astronomer Edmund Halley (1656-1742),

$$\frac{\pi}{6} = \arctan\frac{\sqrt{3}}{3},$$

which is now studied in elementary trigonometry, and another important result from another Briton James Gregory (1638-1675),

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,$$

who obtained a series that today we would write in the following manner

$$\pi = \sum_{k=0}^{\infty} \frac{2(-1)^k 3^{1/2-k}}{2k+1},$$

to calculate π to no less than 71 correct decimals places in 1699. In fact, Sharp calculated 72, but the last one was wrong; he can be forgiven, it is assumed that Sharp had about 300 terms in his series.

We should also mention that in 1667 James Gregory was trying – and failing – to prove that it was impossible to square the circle.

A few years later, in 1706, John Machin (c. 1686-1751), an astronomy professor who became secretary of the Royal Society, released the formula that bears his name today:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

JAMES GREGORY (1638-1675)

It is important not to confuse James Gregory with his nephew, David Gregory (1659-1708), also a mathematician, friend of Newton and one of the proponents of using the symbol π. James is mentioned in the history of astronomy as the inventor of a powerful reflector telescope and for his contribution to mathematical analysis of the series of powers of the trigonometric functions, sin *x*, cos *x*, tan *x*, and their corresponding inverse functions, arcsin *x*, arccos *x* and arctan *x*. Although first discovered by the Indian Madhava of Sangamadrama, the so-called Gregory, or Gregory-Leibniz series, can be written as follows:

$$\theta = \tan\theta - \left(\frac{1}{3}\right)\tan^3\theta + \left(\frac{1}{5}\right)\tan^5\theta - \ldots$$

and converges (remains valid) between $\pi/4$ and $-\pi/4$. Gregory was among the first scholars to suspect that it was impossible to square the circle.

In order to reach this end he followed these steps:

$$\tan\alpha = \frac{1}{5}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{5}{12}$$

$$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{120}{119}$$

$$\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{\tan 4\alpha - 1}{1+\tan 4\alpha} = \frac{1}{239}.$$

And the equality we are after is found in the inverse function, since

$$4\alpha-\frac{\pi}{4}=\operatorname{arc}\tan\ \tan\left(4\alpha-\frac{\pi}{4}\right)=\operatorname{arc}\tan\frac{1}{239}.$$

From this structure the formula is recombined with previous results

$$\operatorname{arc}\tan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots,$$

to produce rapidly converging series with which Machin calculated π to the one-hundredth digit. Without a doubt, the great merit of Machin's formula is that it is trigonometric in form but allows a rapid conversion in series. When we talk, later, of Zacharias Dase, we will see some further curiosities regarding Machin.

Today, Machin-type formulae are established tools and have been developed by many individuals, but Machin was the one who opened the door.

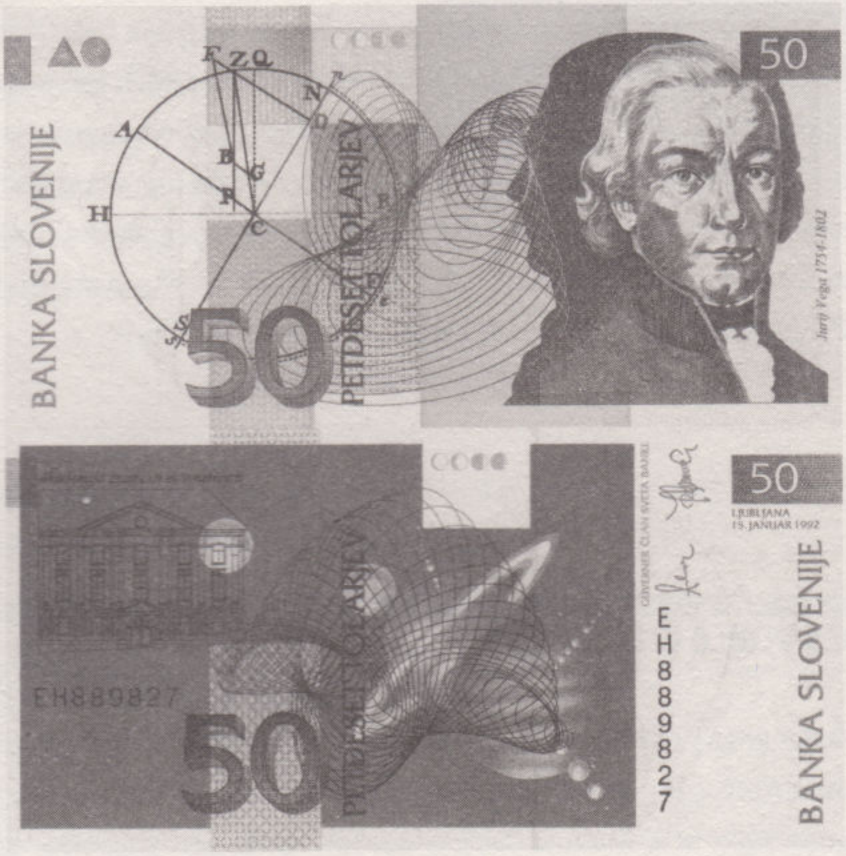
Volume 9 of the first edition of *L'Encyclopédie*, due to the good offices of Denis Diderot, included the work of Thomas Fantet De Lagny (1660-1734), a professor of hydrography and mathematician who calculated π to the tune of 112 digits in 1719. The Frenchman used the same series of powers as Sharp.

AN IRRESISTIBLE CHALLENGE

Thomas Fantet de Lagny (1660-1734), a mathematician born in the French city of Lyon, has won his small place in history with two mathematical exploits: the first was the calculation of 112 correct digits of π, which at the time was the world's best estimate in this peculiar scientific sport. The second, less important, and perhaps apocryphal, happened at the time of his death. It is said that his colleague Maupertuis went to visit his deathbed and found what appeared to be a corpse, showing no signs of life. To be certain, Maupertuis murmured, "What is the square of 12?" perhaps knowing that no numerical soul could resist such a challenge. The alleged corpse almost jumped out of bed to respond with a raucous "144!" ... and then fell dead.

In fact, De Lagny calculated 127 decimal places, but only the first 112 numbers were correct, as shown by the Slovenian soldier and mathematician Jurij Vega or Veha, (1754-1802). In German he is known as Baron Georg von Vega, because at the end of his life he was awarded a knighthood by Imperial Austria, although this did not spare him a commoner's fate: he was killed by muggers. In 1794, Vega used a Machin-type formula, already derived by Euler, to find out 137 exact digits of π, this time without mistakes. The formula was

$$\frac{\pi}{4} = 5 \arctan\frac{1}{7} + 2 \arctan\frac{3}{79}.$$



A 50 Slovenian tolar banknote with the image of Jurij Vega on the front, along with diagrams of geometry and moon phases. On the reverse, to the left of the image of the Solar System is the façade of the Academy of Sciences in Liubliana.

Between 1760 and 1800, co-discoverers revealed several facts about π. Thus, Johann Heinrich Lambert (1728-1777), creator of hyperbolic geometry, showed in 1761 or 1767 (the date is uncertain) that π is irrational, while Adrien-Marie Le–

gendre (1752-1833), concluded that π^2 is also irrational. But perhaps more important is the fact that the great Leonhard Euler (1707-1783), in addition to deducing series after series on π, suggested that it was a transcendental number even before Joseph Liouville (1809-1882) would prove in 1840 that there are transcendental numbers and he had found the first!

JOHANN HEINRICH LAMBERT (1728-1777)

Mathematician, astronomer and physician, this German was the inventor of the first operating hygrometer and photometer. He was also the first to show that π is an irrational number, although this was not his only contribution to mathematics. He studied the hyperbolic functions and connected them with non-Euclidean geometry. Also notable are his contributions to mapping, such as the projections that bear his name. From a very humble origin, he was self-taught yet was not at all modest in assessing his skills. After being appointed to the Berlin Academy by Frederick II, the king asked him once what sciences he had studied. "All of them" was the answer, which was true. The king continued with some sarcasm: "So you also know about mathematics?" "Also" was the truthful answer. Frederick II, a bit annoyed continued: "Who was your teacher?" "I, sir," and this was also true. Already suspicious, the king turned to irony: "Man! Here we are with another Pascal." "At least," was the final answer. His demonstration of the irrationality of π is not complicated to understand and is quite ingenious. Lambert proved (that's the difficult part of the demonstration), using continuous fractions, that if x is a non-zero rational, $\tan x$ is irrational. Since $\tan \pi/4 = 1$ is rational, $\pi/4$, and therefore π, must be irrational.

We have deliberately set aside Euler's career among the decimals of π, as he never achieved a record length and we are not developing an endless list here. And although it is not really what we set out to discuss, we can note that the Swiss mathematician, using his own understanding of the Machin formulae, once calculated 20 decimals of π in one hour!

In 1841, William Rutherford (1798-1871), based on a Machin formula,

$$\frac{\pi}{4} = 4 \arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99},$$

to obtain 208 digits, of which 152 were correct. In 1853 he continued the charge armed with Machin's formula and established a record of 440 digits.

WHAT IS A TRANSCENDENTAL NUMBER?

A number is called algebraic if it is a solution of a polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0,$$

with coefficients a_n, $a_{n-1}, \ldots$, a_1, a_0, all of them rational numbers. In higher mathematics it shows that every constructible number by the rules of the game (i.e., using only ruler and compass a finite number of times) is necessarily an algebraic number. A non-algebraic number is called transcendental. Clearly, then, a transcendental number is not constructible.

Johann Martin Zacharias Dase, or Dahse (1824-1861), occupies a special place in the history of mathematics. A friend of his, L.K. Schulz von Strassnitzky (1803-1852), gave him the following Machin formula

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8},$$

and Dase calculated π in 1844 to 200 digits. The amazing thing is that he did it from memory, in just two months. He was a man with an incredible ability for calculations, a human computer. Gauss himself, the most famous mathematician of his time, recommended that the authorities hire Dase to calculate on their behalf. In fact, there was a grant to develop a list of the divisors of N numbers – $7{,}000{,}000 < N < 10{,}000{,}000$ – and Dase undertook the task but died without ever finishing it. Moreover, he was the perfect example of an *idiot savant*, miraculously endowed for numbers, capable of amazing bursts of memory and rather unwise about everything else. For example, he could multiply two eight-digit numbers in less than a minute. With 100 figures he took longer, about nine hours. He enjoyed an almost photographic memory for counting things, be they sheep, letters or dominoes. In this sense, the writer and scientist Arthur C. Clarke asked in a letter to the paleontologist Stephen Jay Gould, what use could the ability to mentally calculate 200 digits of π have on the evolution of the species. We do not know the answer.

In 1847, the Danish self-taught astronomer and mathematician Thomas Clausen (1801–1885), used two Machin formulae,

$$\frac{1}{4}\pi = 2\arctan\frac{1}{3} + \arctan\frac{1}{7},$$

$$\frac{1}{4}\pi = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

to reach 248 digits exactly. He also erred in his calculations, but did so near the end, as Clausen actually calculated 250 figures.

In 1853, his German colleague Jacob Heinrich Wilhelm Lehmann (1800-1863) reached 261 exact digits and earned mathematical fame and immortality, enough to get a lunar crater named after him. The following year, German Professor Richter reached 330, subsequently 400 and, finally 500 digits.

The amateur English mathematician William Shanks (1812-1882) dedicated his life to the calculation and, as well as other numerical constants, arrived in 1875 to 707 digits of π, a feat that appears in a famous frieze of the Palais de la Découverte in Paris. But this tribute forced the museum to undertake a costly modification of the frieze, built in 1937 because, as demonstrated by D.F. Ferguson and published in 1946 in the magazine *Nature*, only the first 527 were correct. The fact that the number 7 appears too often incited suspicion, as Augustus Morgan (1806-1871) had warned.

Like many others faced with huge calculations, Shanks made mistakes and, lacking references, believed that the digits were correct. We must remember that in his time there were no calculators and computers, and all calculations were done on paper, huge sheets that ended up filled with numbers. At the Palais de la Découverte the corrected figures can now be seen, which is no concession, but a true homage to an explicable error. Now they have found the precise locations where Shanks was wrong, π was calculated not in one go but in phases.

We mustn't forget the input from Ferguson, the last we will mention here who worked at the dawn of the computer age. In 1947 he published 808 digits of π, taking up a year of his time, using a mechanical calculator, patience and the formula

$$\frac{\pi}{4} = 3\arctan\frac{1}{4} + \arctan\frac{1}{20} + \arctan\frac{1}{1985}.$$

In 1882, the German Lindemann poured cold water on the enthusiasm of the seekers of the digits of π when he proved that it is not an algebraic number and thus is not constructible. Lindemann was the one who proved the transcendence of π, and we must point out that such a demonstration, proposed in a memoir with a few dense pages, does not use any geometry. Thus, π has been removed from its geometric frame, and did so on the very day that it demonstrated its transcendence.

Lindemann's original demonstration is based along the same lines as that used some years before by Charles Hermite (1822-1901) to demonstrate the transcendence of the number *e*, another famous constant. Lindemann concluded that a linear combination of powers *e* to algebraic coefficients A_k and algebraic exponents B_k (real or complex),

$$A_1e^{B_1} + A_2e^{B_2} + \ldots + A_ne^{B_n} = 0,$$

was impossible (except for all those $A_k = 0$). As the famous Euler formula can be written

$$e^{\pi i} + 1 = e^{\pi i} + e^0 = 0,$$

it satisfies Lindemann's conditions ($A_1 = A_2 = 1$, $B_1 = \pi i$, $B_2 = 0$) and, therefore, πi can not be algebraic, and π, neither. And if π is not algebraic, it is transcendent. And if it is transcendent, it is not constructible. Then, of course, come other less and less complicated demonstrations, but the first is sufficient to tarnish the lustre of π. Before Lindemann it was known that the transcendence of π implies the impossibility of squaring the circle, and Lindemann's proof forever ended this problem. Squaring the circle was impossible.

Chapter 2

The infinite insignificance and transcendence of π

The face of pi was masked; it was understood that no one could contemplate it and stay alive. But some sharp looking eyes were lurking behind the mask, inexorable, cold and enigmatic. Bertrand Russell

So far we have followed, number by number, the path of π in search of its transcendence, a journey finally ended by Lindemann. We now know that π is transcendental and not constructible, and therefore squaring the circle will forever be an impossible dream.

To understand the true significance of π in the world of mathematics requires an excursion into the stormy world of infinity. This is a world apart, eternally large and equally as intricate, full of questions midway between philosophy and the real world, in short a somewhat unusual place even among the strange universe that is higher mathematics. As a result we will simplify the concepts of infinity as much as possible. We will try to treat it gently so as to not crack open any of the deeper features. Be warned: a mathematical tour into the infinite is not a trivial matter. It will take some effort and any journey has disheartening periods along the way.

With this caveat, we will begin the excursion into the infinite with a question that seems almost absurd: what is a number? To answer that, we will begin by going back to the origins of the idea of number.

Numbers and sets

The basis of almost all numerical ideas are sets, simple collections of things that, for convenience, we represent by putting them between brackets and separating them with commas. For example,

$$A = \{a, b, c, d\}$$

will indicate that set or collection A is made up of a, b, c and d. Instead of letters there may be animals, objects, people, musical instruments or deadly sins; it is all the same. We shall adopt the simplest view possible, which experts describe as *naïve*, and admit that a set is a collection of things or 'elements'.

One set can be coordinated with others. For example, the sets

{*a, b, c*} and {*Napoleon,* ♣, *the author of this book*}

are coordinated, as they can be matched one-to-one without there being a spare element on either side. However, the sets

{*a, b*} and {*Napoleon,* ♣, *the author of this book*}

cannot be coordinated, because however much we try, there will always be one element on the right that does not correspond to one on the other side. The definition of number, therefore, is related to the sets.

A modern way of defining the concept of the number employs a recursive method:

$$\begin{gathered} 1 = \{0\} \\ 2 = \{0, 1\} \\ 3 = \{0, 1, 2\} \\ 4 = \{0, 1, 2, 3\} \\ 5 = \{0, 1, 2, 3, 4\} \\ \dots \\ n = \{0, 1, 2, 3, 4, \dots, n-1\} \end{gathered}$$

We would say that "set A has n elements" when A can be arranged or be matched one-to-one with n. So, we say that the set of players on a football team has 11 elements, or that the set of all the apostles contains 12. The set of 11 is, according to the list we have written,

$$11 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

and no doubt can be matched one-to-one with any set of football players.

And where is zero, which we have yet to define? When we say something has 0 elements? In 'naïve' or elementary set theory, a set is a collection of things. Therefore, among the collections of things there are empty collections, those that do not have anything inside. They are like empty boxes.

However, do not confuse the empty set with nothing, that metaphysical object that is the subject of philosophical debates. The empty set is the one in which there is nothing. It is a set with no elements, but it is not nothing.

To designate such a set (there is only one, since all empty sets are equal), French mathematician André Weil (1906-1998) suggested the Danish letter Ø. Weil knew the Nordic alphabets well because during World War II he was imprisoned in Finland.

We designate the empty set with the letter Ø, and this refers to any set that does not have any elements. Even the most absurd definitions are perfectly valid:

$$\varnothing = \{\textit{flying cows}\}.$$

We define zero as

$$0 = \varnothing,$$

and we say that a set "has 0 elements" if you can put match it up with the set 0. (However, let us not forget that 0 is itself the sole element in set 1).

The number of elements in the set A is indicated by writing the symbol # in front of the set. It is also customary to call this concept *Cardinality of A*, such that

$$\textit{Number of elements in } A = \textit{Cardinality of } A = \#A.$$

In general, we distinguish between finite and infinite sets and usually reserve the concept of 'number of elements' for finite sets. So the finite sets may have, for example, 6, 241 or 123,456,789,012 elements.

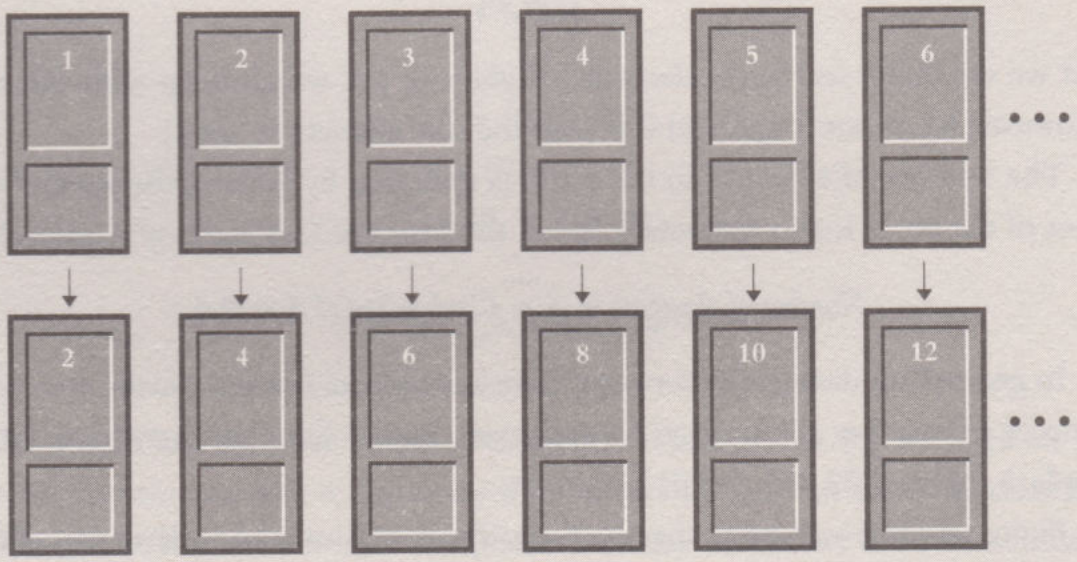
Finite sets have a peculiar characteristic: Their cardinal is larger than any of their constituent parts. If A has, for example, 7 elements, any subset of A would have less than 7. Let's say that

$$A = \{\textit{Snow White's dwarves}\},$$

then $\#A = 7$; any subset or partial group of dwarves B will comply with $\#B < \#A$ and will have less than 7 dwarves. This may seem trivial but is what really distinguishes the finite sets from the infinite. In an infinite set, part of the whole set can have the same cardinality as the original set. Surprising as it may seem, there are effective parts of the whole that have the same number of elements as the whole.

THE HOTEL WITH INFINITE ROOMS

As an example of an infinite set, many mathematicians serve up the paradox of the hotel with infinite rooms, devised by the German mathematician David Hilbert, as follows. There was a hotel run by an owner who is not afraid of overcrowding. The rooms were numbered starting with 1 and continuing in an ascending order. It was high season so the hotel was full and the owner was happy that all the rooms were occupied. But suddenly he got an alarming message from a tour operator: a flood of guests would arrive the next day and they could not be left without accommodation. Moreover, neither could he dislodge the resident guests. The owner, a man versed in mathematics, asked the current residents to move to the room whose number was double that which appeared at his door, as shown in the illustration.



Since he had started with an infinite number of rooms, he could be sure that an infinite-number of rooms were now empty, so the new arrivals could stay in them. The owner of the hotel with infinite rooms continued to operate happily, satisfied with his understanding of infinity.

If we consider the simplest infinite set, that consisting of all positive integers, or the "natural numbers",

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\},$$

which the specialists call $\mathbb{N}$ because it is the initial of *natural*. $\mathbb{N}$ is the term for the set of natural numbers.

We note with surprise that a part of $\mathbb{N}$, the even numbers, can be coordinated with $\mathbb{N}$ itself:

$$\begin{array}{ccc}
0 & \longleftrightarrow & 0 \\
1 & \longleftrightarrow & 2 \\
2 & \longleftrightarrow & 4 \\
3 & \longleftrightarrow & 6 \\
4 & \longleftrightarrow & 8 \\
5 & \longleftrightarrow & 10 \\
6 & \longleftrightarrow & 12 \\
7 & \longleftrightarrow & 14 \\
\vdots & & \vdots
\end{array}$$

Therefore,

$$\#\{\textit{even numbers}\} = \#\mathbb{N}.$$

Thus a subset of something infinite can also be infinite and even have the same cardinality as the whole.

Natural numbers, rational numbers, algebraic numbers

Humans lived for many years, even centuries, without tackling the subject of infinity. From such indifference an exceptional German mathematician broke free. His name was Georg Cantor.

The cardinality of the finite sets is a natural number. Infinite sets have much higher cardinalities, of infinite magnitude. In fact, specialists call them 'transfinite' cardinalities, which means, literally, 'beyond the finite'. The lowest, the first of all transfinite cardinals is #N, the cardinality that Cantor named $\aleph_0$. It corresponds to the cardinality of the set of natural numbers, so

$$\#\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\} = \#\mathbb{N} = \aleph_0.$$

This obscure character needs an explanation. The symbol $\aleph$ (read as *aleph*) is the first letter of the Hebrew alphabet. The subscript zero indicates that you are talking about the lowest of all aleph (aleph zero). Because, of course, there are more aleph numbers, each with its subscript:

$$\aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5, \ldots$$

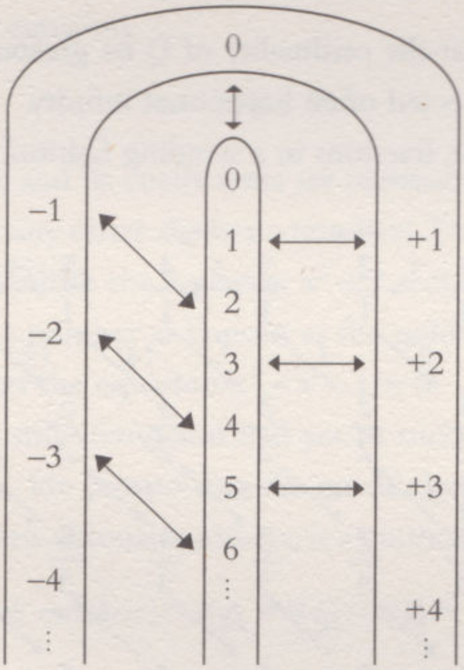
The cardinality $\aleph_0$ in some way counts the sets that can be matched up with N For example, even numbers, odd numbers, multiples of 3, multiples of 5 and an infinity of others. The sets that can be coordinated with N we will call 'numberable', as they can be numbered or counted, as shown opposite:

GEORG CANTOR (1845-1918)

This German mathematician of Russian origin is considered one of the greatest minds in history. He is recognised as the father and creator of modern set theory and transfinite algebra. However, his innovative ideas brought him many powerful enemies, which greatly limited his academic progress. The continuing depression that Cantor suffered – he died in a psychiatric hospital – were perhaps induced by the impossibility of proving some of his assumptions. We now know the answers to some of Cantor's thorniest of mysteries are impossible to find.

$$\begin{array}{ccc@{\qquad}ccc@{\qquad}ccc@{\qquad}ccc}
0 & \longleftrightarrow & 0 & 0 & \longleftrightarrow & 1 & 0 & \longleftrightarrow & 0 & 0 & \longleftrightarrow & 0 \\
1 & \longleftrightarrow & 2 & 1 & \longleftrightarrow & 3 & 1 & \longleftrightarrow & 3 & 1 & \longleftrightarrow & 5 \\
2 & \longleftrightarrow & 4 & 2 & \longleftrightarrow & 5 & 2 & \longleftrightarrow & 6 & 2 & \longleftrightarrow & 10 \\
3 & \longleftrightarrow & 6 & 3 & \longleftrightarrow & 7 & 3 & \longleftrightarrow & 9 & 3 & \longleftrightarrow & 15 \\
4 & \longleftrightarrow & 8 & 4 & \longleftrightarrow & 9 & 4 & \longleftrightarrow & 12 & 4 & \longleftrightarrow & 20 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots
\end{array}$$

But there are many surprises. The infinite set $\mathbb{N} = \{\ldots, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$ is known by mathematicians as the 'set of integers', and $\mathbb{N}$ is a part of the set $\mathbb{Z}$. Every natural number is an integer, that is clear as far as sets are concerned. But what about the cardinality? What is the cardinality of $\mathbb{Z}$? If we look at this diagram



the result is $\#\mathbb{N} = \#\mathbb{Z} = \aleph_0$, so that $\mathbb{Z}$ is also countable.

We will now take a step further and focus on the set of fractions. A fraction has a numerator and denominator expressed in the form a/b. When a is a multiple of b, the fraction a/b is called a whole fraction, and the fraction is also identified with an integer c, the resulting integer, without remainder, resulting from the division of a by b:

$$a/b = c.$$

In fact, several fractions can be identified with the same number:

$$756/378 = 524/262 = 6/3 = 2.$$

However there are obviously others, like 1/2 or 5/3, which are not whole. There are more fractions than integers, since every integer is representable with a fraction. We have, therefore,

$$\mathbb{N} \subset \mathbb{Z} \subset \{\textit{fractions}\}.$$

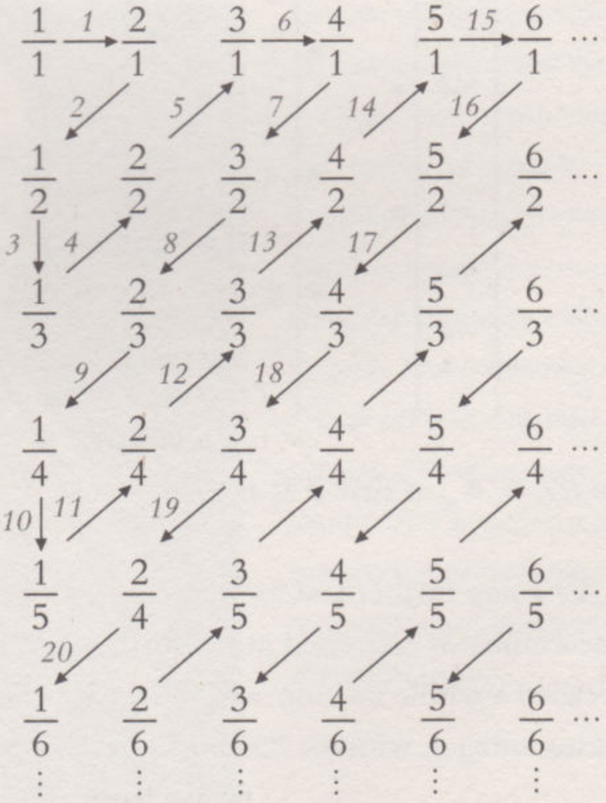
The symbol $\subset$ means 'strictly included in'; it is a variant of the sign < but used for sets, not numbers.

It is customary to denote the set of fractions with the letter $\mathbb{Q}$, and it verifies that $\mathbb{Z}$ is a part of $\mathbb{Q}$. Therefore, putting it another way,

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

One might expect that the cardinality of $\mathbb{Q}$ be greater than that of $\mathbb{Z}$, but it as suspected that the unexpected often happens at infinity.

Cantor 'numbered' the fractions in a winding fashion, devising a chart like this:



There is no doubt that this comprises all the fractions, since in each row are all the possible numerators, and in the columns, all the denominators. If we look for a number of the form a/b, it is easily found in column a and row b. And there is no doubt that each fraction (i.e., every rational number) corresponds to a sequence of arrows, one after the other, leading to it. We only have to number the arrows (1, 2, 3, 4, 5 ...) to reach the result:

$$\#\mathbb{Q} = \#\mathbb{Z} = \#\mathbb{N} = \aleph_0.$$

We will take one more step. It is said that a number is *algebraic* when it is the solution to a polynomial equation,

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

with coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$, all of them rational numbers. There is a wealth of algebraic numbers, and, in fact, every rational number is algebraic. If you take any rational number, a/b, the equation

$$x - a/b = 0$$

has as a solution $x = a/b$, and its coefficients are rational: $a_1 = 1$ *and* $a_0 = -a/b$.

Although there are many other algebraic numbers. The number $\sqrt{2}$, for example, is also irrational and fulfills the equation $x^2 - 2 = 0$; therefore, it deals with an algebraic number. Even a number as famous as the golden ratio, Φ, is an algebraic number because it satisfies the equation $x^2 - x - 1 = 0$.

In 1874, Cantor was still young and had yet to succumb to the mental illness that plagued his later life. He proved that the set of algebraic numbers, which we call A, which includes even rational numbers, is a countable set. Therefore,

$$\#\{\textit{algebraic numbers}\} = \#\mathbb{A} = \#\mathbb{Q} = \#\mathbb{Z} = \#\mathbb{N} = \aleph_0.$$

And each is a set strictly greater than the following:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A}.$$

The real numbers arrive

The universe of numbers is very large. So far we have seen only a portion of the numeric world, which also deals with a numbered part that can be counted.

Perhaps the best way to talk about numbers is to characterise them by their decimal expression. So we think about the set of all decimal numbers and explore the land inch-by-inch.

Generally speaking, a decimal number such as

$$34{,}658.124796$$

is nothing more than a way to write

$$\mathbf{3}\cdot 10^4+\mathbf{4}\cdot 10^3+\mathbf{6}\cdot 10^2+\mathbf{5}\cdot 10^1+\mathbf{8}\cdot 10^0+\mathbf{1}\cdot 10^{-1}+\mathbf{2}\cdot 10^{-2}+\mathbf{4}\cdot 10^{-3}+ \\ +\mathbf{7}\cdot 10^{-4}+\mathbf{9}\cdot 10^{-5}+\mathbf{6}\cdot 10^{-6},$$

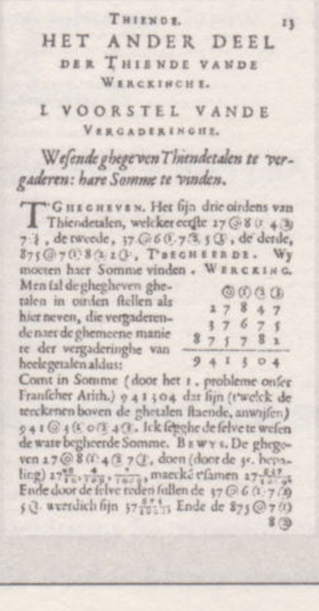
with the numbers listed to the left of the decimal point indicating positive powers of 10, and the right, negative powers. We remember that

$$a\cdot 10^{-n}=\frac{a}{10^n}.$$

Decimals are the expression of numbers in base 10, with positional notation and zero. These are just a method of writing, but magnificent, a great achievement of mankind!

SIMON STEVIN (1548-1620)

This Dutch scientist, born in Bruges (which is now in Belgium), worked in diverse fields such as military engineering, music, physics, mathematics and accounting. He also went down in history as the inventor of double entry bookkeeping, an unsung contributions of arithmetic to the advancement of civilisation. But Stevin's mathematical contribution was even more important; in *De Thiende (The Decimals)*, he created decimal notation, but it was so complicated to put on paper that later systems, such as those proposed by John Napier, won the day.



THIENDE. 13

HET ANDER DEEL
DER THIENDE VANDE
WERCKINGHE.

I VOORSTEL VANDE
VERGADERINGHE.

Weſende ghegeven Thiendetalen te vergaderen: hare Somme te vinden.

T'GHEGHEVEN. Het ſijn drie oirdens van Thiendetalen, welcker eerſte 27⓪8①4②7③, de tweede, 37⓪6①7②5③, de derde, 875⓪7①8②2③. T'BEGHEERDE. Wy moeten haer Somme vinden. WERCKING. Men ſal de ghegheven ghetalen in oirden ſtellen als hier neven, die vergaderende naer de ghemeene maniere der vergaderinghe van heelegetalen aldus:

⓪ ① ② ③
2 7 8 4 7
3 7 6 7 5
8 7 5 7 8 2
9 4 1 3 0 4

Comt in Somme (door het 1. probleme onſer Franſcher Arith.) 941304 dat ſijn (t'welck de teeckenen boven de ghetalen ſtaende, anwijſen) 941⓪3①0②4③. Ick ſegghe de ſelve te weſen de ware begheerde Somme. BEWYS. De ghegeven 27⓪8①4②7③, doen (door de 3e. bepaling) $27\frac{8}{10}, \frac{4}{100}, \frac{7}{1000}$, maecken t'ſamen $27\frac{847}{1000}$. Ende door de ſelve reden ſullen de 37⓪6①7②5③ weerdich ſijn $37\frac{675}{1000}$; Ende de 875⓪7①8②

A page from De Thiende *shows an example of Stevin's decimal, unsuitable for frequent use. Units are indicated with a small circle enclosing a 0; tenths, with another small circle with a 1; hundredths, with a 2, and so on.*

A decimal number can have an end or may not have one. Here is an example of each:

$$1.2345678901011121314151617181920212223242526...$$

$$127.789564.$$

The first is a decimal with infinite figures. The second also has them, since

$$127.789564 = 127.7895640000000000000000000...$$

In fact, we can even find another more 'sophisticated' way to write

$$127.789564 = 127.7895639999999999999999999...$$

However, in these cases we will refer to a decimal number with finite digits or of finite expression. The most simple decimal numbers are the natural numbers ($\mathbb{N}$), because they are positive and lack digits after the decimal point. Then follow the whole numbers ($\mathbb{Z}$), which may be negative, but also do not have digits after the decimal point. The rational numbers ($\mathbb{Q}$) include them both and have a peculiar decimal expression: the digits of a rational number have a period, i.e., they repeat after a certain point. Recall that rational numbers are fractions, all of them in the form *a*/*b*, with *a* and *b* being integers. To go from this form to a decimal expression we must divide *a* by *b*, and what happens? As the remainder is a maximum of *b*, after *b* divisions the numbers are repeated again and again. This circumstance can be seen perfectly in the most basic fractions, such as the following:

$$\frac{11}{7} = 1.571428571428571428...$$

The period or string of numbers that is repeated is always 571428. Sometimes, the period is huge – the decimal number certainly does not cease to have infinite numbers, but they are infinitely repeated.

At this point, the obvious question is: if periodic decimals identify rational numbers, what about the decimals that are not periodic? It's simple: If they are not periodic, there are not rational numbers, but 'irrational'.

THE DIAGONAL DEMONSTRATION

Cantor's reasoning that led to the demonstration of the uncountability of all real numbers (from $\mathbb{R}$) has gone down in history as a sign of genius. It is both original and comprehensible. It has become so famous it even boasts its own name, the diagonal procedure, diagonal argument or diagonal demonstration. Let's see what the term 'diagonal' means.

We will proceed to make what is known in mathematics as 'proof by contradiction', which begins as an assumption or hypothesis and shows that it leads to an impossible conclusion. Therefore, this hypothesis can not be true.

We will assume as a hypothesis (which we will see is false) the countability of the decimal (or real numbers). Now we do not even assume the countability of everything in $\mathbb{R}$, only the real numbers of interval x (0,1), i.e. the real numbers $0 < x < 1$, which is a tiny part of $\mathbb{R}$.

In the present case, we start from the assumption that we have numbered and listed the real numbers one below the other, though not necessarily in order, as shown here:

$$\begin{aligned}
&\mathrm{N} \leftrightarrow \text{real numbers}(0,\ 1)\\
&1 \leftrightarrow 0.835987\ldots\\
&2 \leftrightarrow 0.250000\ldots\\
&3 \leftrightarrow 0.559423\ldots\\
&4 \leftrightarrow 0.500000\ldots\\
&5 \leftrightarrow 0.728532\ldots\\
&6 \leftrightarrow 0.845312\ldots\\
&\quad\ldots\\
&n \leftrightarrow 0.r_1r_2r_3r_4r_5\ldots r_n\ldots
\end{aligned}$$

There are many irrational numbers, from $\sqrt{2}$ and all the combinations of radicals, such as $\sqrt{\frac{3\sqrt{5}+1}{2}-\sqrt[7]{3}}$, to universal constants such as π. The next logical question to ask is: "How many irrationals are there?"

Let's use $\mathbb{R}$ as the name the set of all the decimal numbers, that is, to the sum (or union, $\cup$) of all the rational and irrational numbers:

$$\mathbb{R} = \{rational\} \cup \{irrational\}.$$

We know that the first of these sets, $\mathrm{Q} = \{rational\}$, is countable; because, Cantor demonstrated that the total set, $\mathbb{R}$, "is not countable". Therefore, the set on the

The list must include all decimals between 0 and 1, so it would be false to construct a decimal *n* that is not on the list. And that was Cantor's basis for inventing a new decimal number *D*,

$$D = 0,\ d_1\ d_2\ d_3\ d_4\ d_5\ \ldots\ d_n\ldots,$$

that was not in the list. This defined for each *n* a number d_n different from that in row *n*, column *n*.

Is *d* different from the decimal number corresponding to 1? Yes, because by construction, d_1 is different from the first decimal digit of that number.

Is *d* different from the second decimal number in the list? Yes, because the number d_2 is different from the second digit of the second decimal.

Is *d* different from the third decimal number in the list? Yes, because the number d_2 is different from the third digit of the third decimal.

And the same is true, by construction, for the fourth, the fifth... the *n*th decimal:

$$d_n \neq r_n.$$

D is different from all the decimal numbers on the list, and then is not on the list. But had we not assumed that they were all there? There is a contradiction with the assumption of departure, stating that the decimals were all numbered and listed one below the other. In fact, we neither had nor will we ever have them, as it has been proven that the set of all decimals is uncountable.

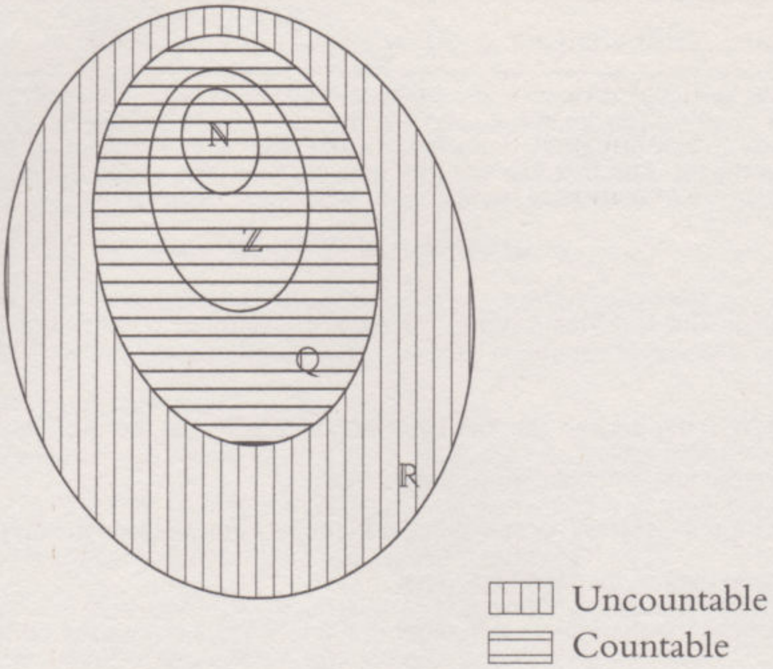
$$\#\ \mathbb{R} > \aleph_0.$$

right also cannot be. Otherwise, ℝ would form the union of two countable sets and, consequently, would also be countable.

We have finally found something uncountable, a set ℝ that we cannot count, we cannot number and that is certainly a greater infinity than all infinite sets which we had encountered so far.

The set ℝ is known as 'the set of real numbers'.

In the most basic set theory, studied in school today, there are often diagrams used that graphically summarise all that has been exposed so far:



The sets ℕ, ℤ and ℚ, internal to ℝ, are countable, while ℝ is not. We could say, slightly abusing the language, that almost all numbers are irrational, with the exception of the rational numbers, forming a smaller infinity, a countable infinite. The number π is irrational, as demonstrated by Johann Heinrich Lambert (1728-1777) in the 1760s, and belongs, therefore, to the uncountable majority that encompasses almost all decimal numbers. From this point of view, π is hardly exceptional; it is a real number and is irrational, like almost all of them.

Algebraics and transcendents

We have previously spoken of algebraic numbers. Remember that:

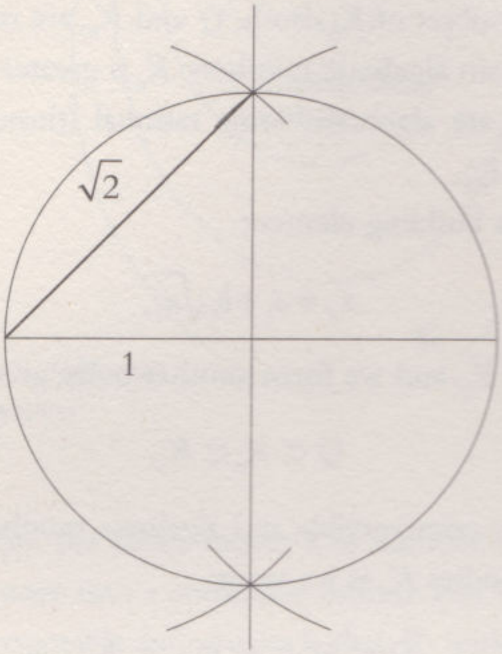
1. The algebraic numbers are those that can solve any polynomial equation with rational coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0;$$
$$a_n, a_{n-1}, \ldots, a_1, a_0 \text{ rational numbers.}$$

1. The algebraic numbers form a countable infinity.

Why do they reappear now? The reason is that all truly geometric figures can be constructed with just a ruler and compasses in a finite number of steps.

The requirement to use only the purist Greek ruler and compasses in all geometric constructions led to some special segments (and therefore some special numbers) which were described as 'constructible'. Consider a single number, such as $\sqrt{2}$. It is a Euclidean number, which can be plotted perfectly with ruler and compasses, as shown in this figure:



This is the first irrational number that the Greeks found, and is directly responsible for the name. As well as being irrational, $\sqrt{2}$ is constructible and algebraic. As we have seen, $\sqrt{2}$ is the solution of the second-degree polynomial equation $x^2 - 2 = 0$.

Constructible numbers are all algebraic, and let's see why. When we are considering a geometric shape drawn with a ruler and compasses, the most we can build are numbers of the form

$$x_1 = a_0 + b_0\sqrt{x_0},$$

where a_0, b_0 and x_0 are rational. We will save this demonstration, not because it is complicated, but because it is laborious. The number x_1 is algebraic, it is the solution to a quadratic equation with rational coefficients, namely

$$x^2 - 2a_0x + a_0^2 - b_0^2x_0 = 0.$$

This expression is an equation with rational coefficients, as they all are: 1, as much as $-2a_0$ and $a_0^2 - b_0^2 x_0$ are elements of Q. All numbers like x_1 form what is called 'body', which we will call K_0 and which verifies that

$$\mathbb{Q} \subset K_0.$$

That means that Q is a subset of K_0. Both, Q and K_0, are made up of constructible numbers, but only contain algebraic numbers. K_0 is greater than Q and contains it. All the numbers of K_0 are algebraic, some rational (those in Q) and others not (those in K_0 but not in Q).

Now we choose as a building element

$$x_2 = a_1 + b_1\sqrt{x_1},$$

with the a_1, b_1 and x_1 of K_0, and we form another body, greater still:

$$\mathbb{Q} \subset K_0 \subset K_1,$$

which is also formed by constructible and algebraic numbers. It is evident that we can fabricate as many bodies K_n as we want:

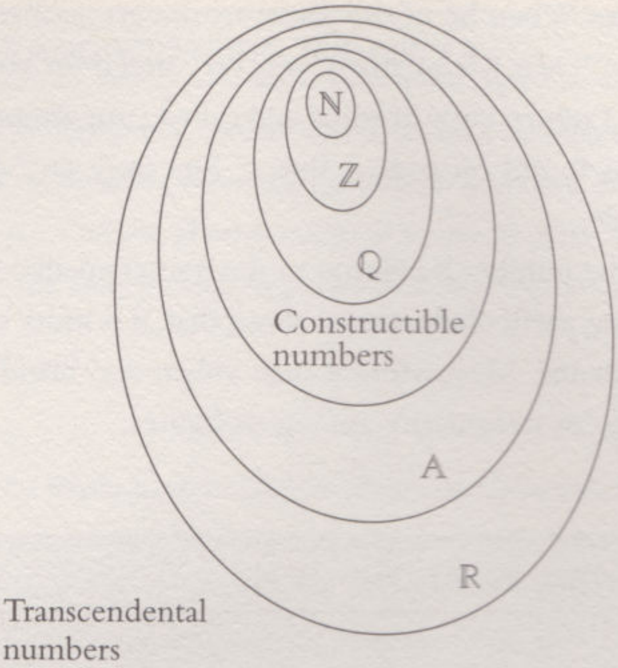
$$\mathbb{Q} \subset K_0 \subset K_1 \subset \ldots \subset K_n.$$

In general, the geometric constructions that they characterise are second-grade equations (also called quadratic equations) that may follow each other, but, even if they do, eventually result in the construction of other numbers, which ultimately are also algebraic.

Thanks to analytic geometry invented by René Descartes in the 17th century, all geometric constructions are translatable into quadratic equations; a more complicated construction may result in a chain of second grade-equations, one inside the other.

But however long the chain, the result is always a constructible number that, when solving a second-grade equation with constructible coefficients, is algebraic. Using geometric constructions we never leave the realm of constructible numbers, and therefore, the algebraic realm. Every constructible number is algebraic.

We will not begin the rigorous and detailed demonstration of this fact, because we would need to use the tools of the Galois theory taken from higher mathematics. The facts can be represented graphically, as in the diagram opposite:



So taking into account the entire universe of numbers, from the frontier of the algebraic numbers we move into a countable infinity. But we already know that ℝ is a much larger, non-countable infinity, such that ℝ without the algebraic numbers, covers 'almost everything' because ℝ also has a transfinite number of non-countable components.

Mathematicians call the non-algebraic numbers (remember that these are all real numbers minus those that are algebraic, i.e., all of ℝ except 𝔸) 'transcendental numbers', since, according to Euler, such numbers are "beyond the power of algebraic methods". This definition has philosophical connotations, but nevertheless is utterly precise: a transcendental number is simply one that is not a solution of any polynomial equation with rational coefficients. All transcendental numbers are irrational and non-countable; their cardinalities are greater than $\aleph_{0.}$

And what relation does this have to π? Well π is not only irrational, but is also transcendent, as Lindemann demonstrated in 1882. If π is transcendent, it is not algebraic, it is not constructible and, therefore cannot be built in a finite number of steps using only ruler and compasses. So the search, begun by the ancient Greeks, for a method to square the circle was over. But the arcane pursuit of the unattainable is not over: even today distinguished mathematicians are still sent 'proofs' of squaring the circle. And there are those who have pre-cooked and prepared a response to deter the would-be discoverers of pursuing impossible demonstrations. Often such proofs that pass across the desk of a mathematics professor are given to a pupil as an exercise. When he of she discovers the error (there is always an error) they send this reply: "Dear friend, thank you very much for your demonstration of squaring the circle. I return your original and called your attention to the first error we encountered. As found on page..., line..., Sincerely, etc." It is a polite way to rebuff the impossible.

In conclusion, the number π belongs to the transcendental numbers and, apparently, there is nothing particularly special about that. It is more unusual for a number to not be transcendental. Meanwhile π is so vulgar and bland that no one has yet found any regularity or irregularity among its figures.

TRANSCENDENTAL RELATIONS OF PI

The number e is the base of natural logarithms, and its value is 2.71828... and, after π, is the most frequently used mathematical constant. We know, of course, that

$$\pi + e = 5.859874482\ldots,$$

but whether or not this number is or is not transcendental is ignored. Curiously, it is known that one of the two numbers $\pi + e$ or $\pi \cdot e$ are transcendental, but not which one. It is also not known whether π^{π} is transcendental.

However, e^{π} is transcendental, as demonstrated thanks to the mathematical theorem of Alexandr Gelfond (1906-1968) and Theodor Schneider (1911-1988). Although the same can not be said for π^{e}; in fact, we do not even know if it is rational or irrational.

Also transcendental are $e^{\pi\sqrt{n}}$ (if $n \neq 0$), $\pi + \ln 2$ and $\pi + \ln 2 + \sqrt{\ln 3}$.

It is not known if $\pi + e$ or π / e are irrational in terms of their transcendence. We know that if they are algebraic, the eventual polynomials are sure to have as their solution at least grade 8, with middle order coefficients 10^{9}. It is not mathematically relevant, but rather compelling.

Squaring the circle

After reviewing the nature of π and finding it to be transcendental, it is clear that to attempt to square the circle is pointless. However, in the days before Lindemann there were those who were very much engaged in that search, always in

ARISTOPHANES AND SQUARING THE CIRCLE

The Greek dramatist Aristophanes (c. 446-to 386 BC) spoke of squaring the circle in a rather funny code in one of his comedies, which was run through with satire. In *The Birds*, released in the year 414 BC, some citizens of Athens, tired of the bustle of the metropolis, decided to found a city in the sky. Several architects and planners offer their services to the protagonist, Pithetaerus.

[METON]: I've come to...

[PITHETAERUS]: Has my misery no end! Come to what? What is your plan?

[METON]: I have come to survey this Air of yours, and parcel it into plots.

[PITHETAERUS]: For heaven's sake! Who are you?

[METON]: Who am I? I am Meton, known throughout all of Greece.

[PITHETAERUS]: Very well. And what is this?

[METON]: This is a wand to measure the Air. I will explain it to you. The space of the Air is like a large furnace; thus, I use my bent wand here, and place the compass here... Do you understand?

[PITHETAERUS]: Not a word.

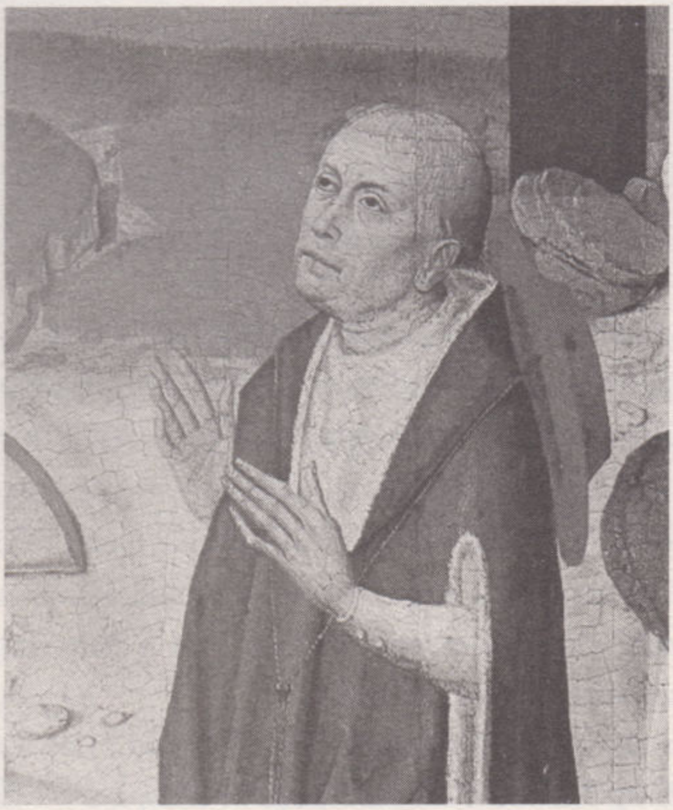
[METON]: With this other ruler I draw a straight line, inscribing a square in the circle and I place the forum in its centre. All of the streets will lead to it, such as it is with the centre of a star, which although circular, sends forth straight rays in all directions.

[PITHETAERUS]: By the gods, this man is a regular Thales!

good faith, and with intelligent approximations. Most π decimal hunters were actually square seekers; they were affected by what one joker called *morbus cyclometricus*, the infectious disease of wanting to square the circle. A description of the average 'squarer' would be male, mature, oblivious to the meaning of the word impossible, with little mathematical knowledge, convinced of the importance of the problem and of earning a large reward, illogical, lonely and if that wasn't enough, also prolific cataloguers and writers. Not a flattering portrait by any means but one that approximates to reality.

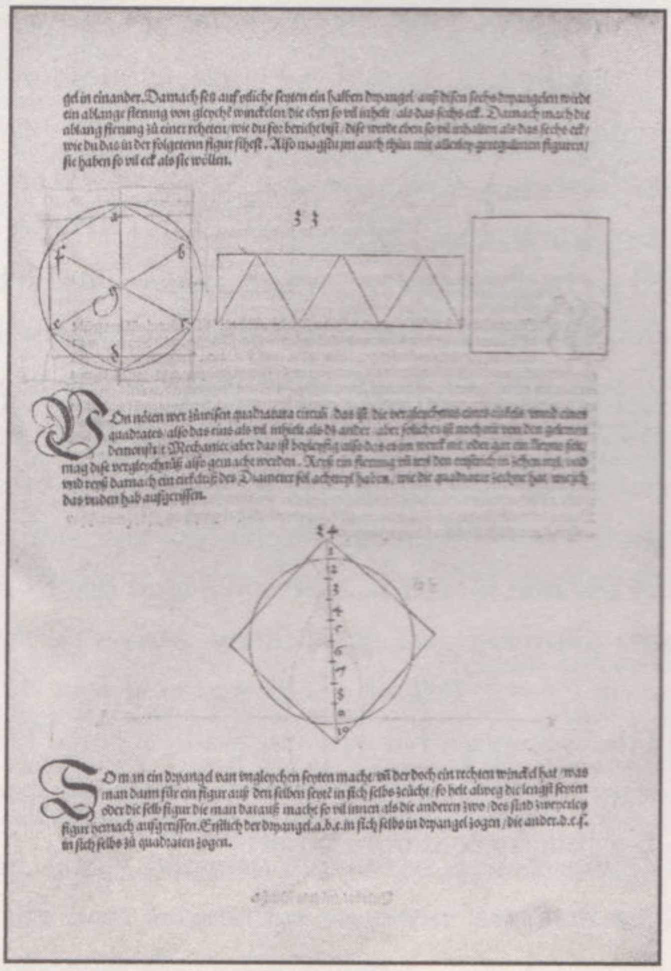
The Roman philosopher Boethius (c. 480-524) said in his *Liber circuli* – before his execution by King Theodore on the charge of conspiracy – that he had squared the circle but the proof was too long to reproduce on paper in its entirety. This phrase, later used by Fermat in reference to his famous last – and for many years unproven – theorem, together with the indisputable fact that the squaring is impossible mark Boethius out as either mistaken or fraudulent.

Nearer our own time, we find the celebrated German Cardinal Nicholas of Cusa (1401-1464); his great intellectual stature was quoted even by Kepler and Cantor, given that he pondered very advanced ideas about infinity. He was an excellent linguist, jurist, philosopher, astronomer and more of a numerologist than a mathematician. As a geometer he tried (and according to him, succeeded) to square the circle. However, his contemporary Johannes Müller von Königsberg (1436-1476), who went by the Latinised pseudonym Regiomontanus (or Regiomontano), a better mathematician than the clergyman and a great admirer of Archimedes, refuted these assertions and showed that there was no such square in his work *De Cuadratura Circuli*. Yes it is true, however, that Nicholas of Cusa's approximation of π (which he gave as an exact figure) is quite good: 3.1423... Regiomontanus gave an approximate value to π of 3.14243.



Cardinal Nicolas de Cusa said he had managed to square the circle.

In 1525, the great artist Albrecht Dürer (1471-1528) also tried to square the circle, but flagged up that its construction was only approximate.



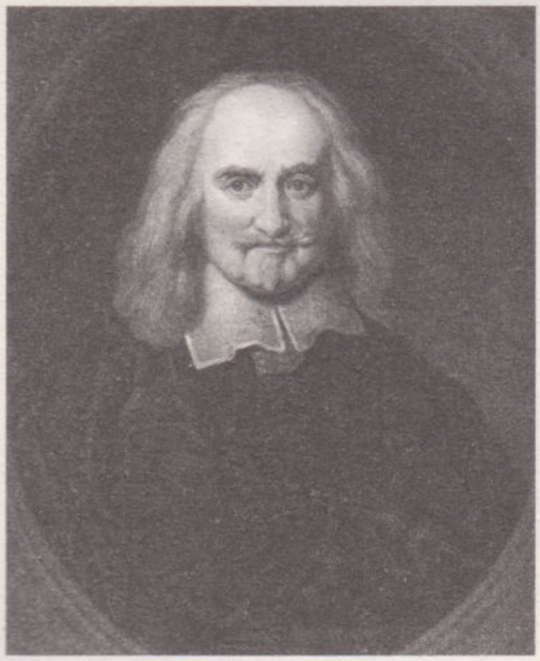
A page from Underweysung der Messung *by Dürer, showing his approximate squaring of the circle.*

A little later, in 1585, Adriaan Anthonisz (circa 1543-1620), father of Adriaan Metius (1571-1635), calculated that π was between 377/120 and 333/106. The son had an easy go at squaring the circle: it was enough to take an average of the numerators and denominators:

$$\pi = \frac{\frac{1}{2}\left(377+333\right)}{\frac{1}{2}\left(120+106\right)} = \frac{355}{113} = 3{,}14159292\ldots$$

It is a good approximation, but as expected did not square the circle...

Perhaps the best known story of the squares is the one that involved Thomas Hobbes (1588-1679), renowned philosopher and champion of empiricism, and John Wallis (1616-1703), an eminent British mathematician. Apparently, Hobbes, a very intelligent man but lacking a specialised education in geometry, proclaimed in *De Corpore*, in 1655, that he had squared the circle, in addition to other achievements such as corrections of several curves. This cannot have been true, and Wallis, in his work *Elenchus Geometriae Hobbianae*, discovered several errors and slipped in some harsh but true opinions about the geometric talent of Hobbes. It must be said that Wallis professed to be Presbyterian, which made him doubly odious to the distinctly high-church Hobbes. Hobbes's mathematics was poor: he only fell under Euclid's spell at 40, but, after all, many other philosophers have been mediocre mathematicians and still proved to be relevant. For example, Marx said in the middle of the 19th century that dialectic materialism was could be deduced from the second-grade equation. For Hobbes, the trouble was that not only could he not admit his errors, but became locked in a personal feud played out in his writing. The titles, ever more poisonous, are quite funny: *Markes on the Absurd Geometry, Rural Language, Scottish Church, Politiks, and Barbarism of John Wallis.* The disputees did not spare each other's feelings. For example Wallis accused Hobbes of plagiarising his contemporaries: "... when something real is included among your things it is not really yours, but things taken from others."



Thomas Hobbes (left) and John Wallis were involved in a long-running dispute, trading insults through their work. Squaring the circle was at the centre of their feud.

JOHN WALLIS (1616-1703)

The famous symbol for infinity, ∞, will serve to present the excellent English mathematician, as it was he who introduced it. Linked to the Royal Society, Wallis worked on deciphering messages and, more than that, on issues related to the height of scientific fashion of the time, infinitesimal calculus, to which he provided new and interesting concepts. His most notable creation is in the field of the series and, more specifically, in infinite products series, which gave us a beautiful and useful formula:

$$\prod_{n=1}^{\infty}\frac{(2n)(2n)}{(2n-1)(2n+1)}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\ldots=\frac{\pi}{2}\cdot$$

Wallis was also a great mental calculator, perhaps because he suffered from insomnia. He was also a grammarian, an uncommon thing, he devoted a lot of effort to teaching the deaf.

Grégoire de Saint-Vincent (1584-1667), a Belgian Jesuit to whom we owe, among other things, polar coordinates, invented a new system close to the concept of integration that correctly found the squaring of the hyperbola – and claimed to have found the squaring of the circle. His contemporaries took it with understandable scepticism and, finally, Huygens discovered the inevitable error in his reasoning. We mention this here because his otherwise excellent work showed a lot of interesting and mathematically certain things.

A typical example of squaring the circle is represented by Jacob Marcelis (1636-to 1714), a soap manufacturer who stated that

$$\pi=3\frac{1008449087377541679894282184894}{6997183637540819440035239271702},$$

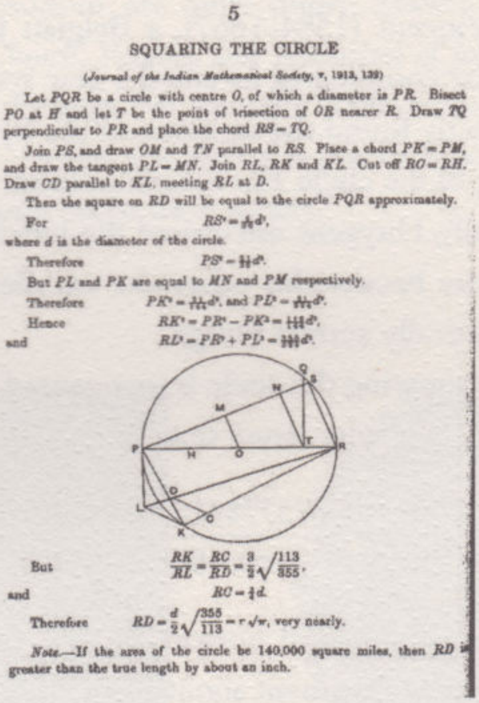
In his anthology of mathematical horrors *A Budget of Paradoxes,* De Morgan made somewhat uncharitable comment about Marcelis: "I hope his soaps are better than his values of π."

The follies accumulate as we progress through time and thus we arrive at Malthulon, who in 1728 claimed to have unravelled, simultaneously, perpetual motion and squaring the circle. Also, he offered a reward to anyone who could disprove a single step; a pathetic and misplaced confidence. It goes without saying that his work was shown to be false, and Malthulon had no choice but to pay.

No wonder that in 1753, the French Academy decided not to undertake a review of any more demonstrations of squaring the circle. Maybe they were horrified by the growing number of requests and the cost to review them, or maybe the academics wanted to save themselves from a certain group of tenacious but flawed characters.

The parade of squaring did not end with Lindemann, but at least it was now known that their demonstrations were always wrong. We have to exclude those who, like Srinivasa Ramanujan (1887-1920), knew well that it was impossible to square the circle, but looked for approximate constructions with amazing accuracy. With one of Ramanujan's construction we find a value of

$$\pi \approx \sqrt[4]{9^2 + \frac{19^2}{22}} = 3.1415926525826\ldots$$



5

SQUARING THE CIRCLE

(*Journal of the Indian Mathematical Society*, v, 1913, 132)

Let PQR be a circle with centre O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord $RS = TQ$.

Join PS, and draw OM and TN parallel to RS. Place a chord $PK = PM$, and draw the tangent $PL = MN$. Join RL, RK and KL. Cut off $RC = RH$. Draw CD parallel to KL, meeting RL at D.

Then the square on RD will be equal to the circle PQR approximately.

For $RS^2 = \frac{5}{36}d^2$,

where d is the diameter of the circle.

Therefore $PS^2 = \frac{31}{36}d^2$.

But PL and PK are equal to MN and PM respectively.

Therefore $PK^2 = \frac{31}{144}d^2$, and $PL^2 = \frac{31}{324}d^2$.

Hence $RK^2 = PR^2 - PK^2 = \frac{113}{144}d^2$,

and $RL^2 = PR^2 + PL^2 = \frac{355}{324}d^2$.

But $\frac{RK}{RL} = \frac{RC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}}$,

and $RC = \frac{3}{4}d$.

Therefore $RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}$, very nearly.

Note.—If the area of the circle be 140,000 square miles, then RD is greater than the true length by about an inch.

Ramanujan's approximate construction of squaring the circle. The error is only 0.0000000010072!

Chapter 3

π and probability

Probability theory is just, basically, common sense reduced to calculation.
Marquis de Laplace

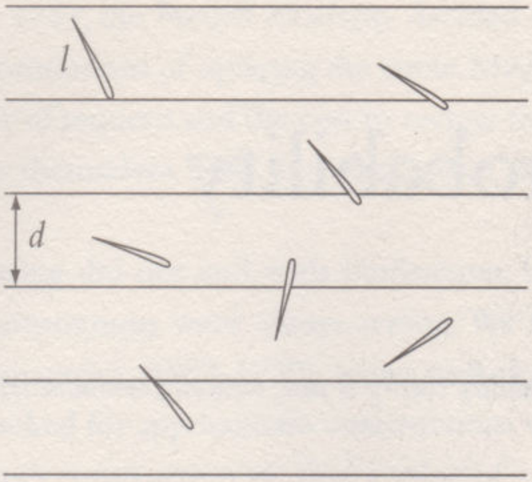
In speaking of probability it seems that we are delving deeper into a concept far removed from π; nothing could be further from the truth. From the outset, $0.6079271018... = 6 / \pi^2$ is the probability that two randomly chosen integers are relatively prime, announced by R. Chartres in 1904. Furthermore, $\pi^2/6 = \zeta\ (2)$, which establishes a curious link between π and Riemann's mysterious function ζ (zeta). It also establishes an unexpected relationship between π and probability theory, as the magic number is an unexpected guest here. Finally, it establishes another link between π and prime numbers.

On one occasion, Augustus de Morgan posed a mathematical question to an insurance salesman, a specific calculation involving the probability that a group of people had to stay alive after a certain time, and as often happens with the probability theory, who should show up but the number π. The insurance salesman, convinced that De Morgan had made a mistake, let him know. How was it possible that π could appear when talking about insurance? What was π doing there? And yet, De Morgan had been right: there really was a link between life expectancy, the policies and π; it is called 'normal distribution'.

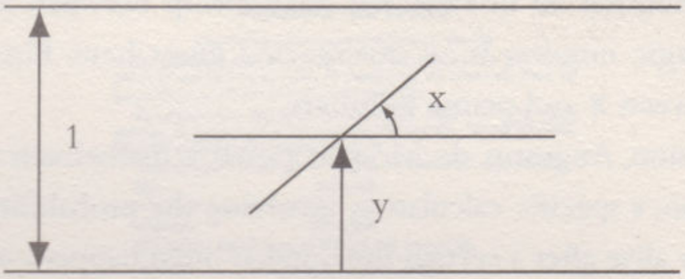
The purpose of this chapter is to show these hidden links. The story begins with the arrival of a French nobleman, Comte de Buffon, in the world of mathematics. Buffon had the idea of studying the mathematical behaviour of a needle that falls flat, without sticking, on a series of parallel lines.

A needle in a haystack

Try this yourself. Draw a series of equidistant parallel lines on paper. Then throw a needle at random. How often does the needle land across one of the lines?



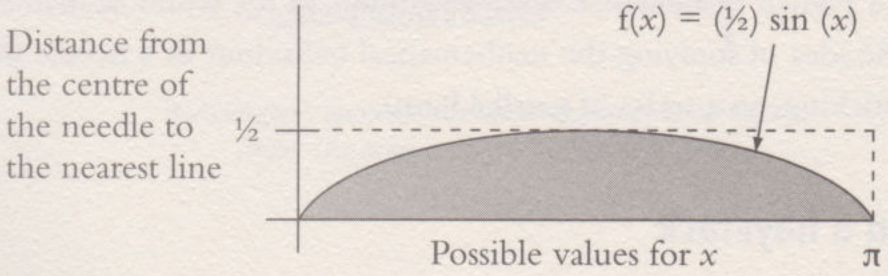
The simplest case occurs when the length l of the needle is equal to the distance d between the lines. Let's call y the distance from the midpoint of the needle (its supposed centre of gravity) to one of the lines. If we assume that $d = l = 1$, the calculations are a bit more simple. Let's now call x the angle formed between the needle and the horizontal axis, measured in radians, since we will use integrated analysis.



Using a little elementary geometry we see that when the inequality

$$y \leq \tfrac{1}{2} \sin x,$$

the needle will cut a line. Thus we have a starting point. The following diagram is a graphical representation of the function $y = \tfrac{1}{2} \sin x$:



GEORGES-LOUIS LECLERC, COMTE DE BUFFON (1707-1788)

Buffon was a universal scholar, best known as a naturalist.

This French scientist left his mark in many different disciplines. He was a naturalist, writer, biologist, botanist, cosmologist and mathematician. His masterpiece was *Histoire Naturelle, Générale et Particulière (Natural History, General and Particular)*, presented in no less than 36 volumes and 8 appendices. In his role as a cosmologist, Leclerc's most notable contribution is his hypothesis about the age of the Earth, which he calculated based on the rate at which a ball of red-hot iron cooled, and which threw-up a serious disagreement with the official doctrine of the Church. He translated Newton into French and contributed to the theory of probability with a charming self-published title, *Essai d'Arithmétique Morale (Essay on Oral Arithmetic)*, containing, among other things, the famous probability study of a needle falling on a set of parallel lines.

To evaluate the size of the shaded area, which is one in which $y \leq ½ \sin x$, we must calculate an integral:

$$\int_0^\pi \frac{1}{2} \sin x \, dx = 1$$

The value of the entire area of the rectangle is $\pi / 2$, and the probability that the needle falls on the line is the quotient between one area and the other:

$$\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \approx 0.6366197...$$

And that is where π appears.

The problem can also be solved if $l \neq d$; if $l < d$ we get a probability of $2l/\pi d$, and if $l > d$, the probability is

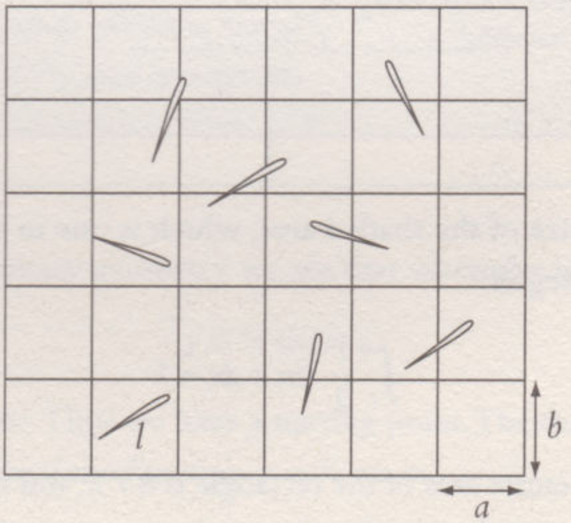
$$\frac{2l}{d\pi} - \frac{2}{d\pi}\left(\sqrt{l^2 - d^2} + d \sin^{-1}\left(\frac{d}{l}\right)\right) + 1.$$

In this case we must resort to calculating a double integral.

Based on this procedure, the calculation of π has been attempted, but the results have not been very satisfactory. In fact, small irregularities in the needle could cause serious errors, so this method is not recommended. It is better to simulate the experiment virtually throwing needles on linear lattices in cyberspace; there are computer programs that do it this way.

And a needle in a grid

Buffon-Laplace is the name of a more complicated version of the previously problem, studied by Laplace in 1812, in his *Traité Analytique des Probabilities*, in which the needle does not fall on equidistant parallel lines, but on a grid of horizontal and perpendicular lines. Each cell in the grid measures *a* along one side and *b* along the other ($a \neq b$), and we assume that the needle is shorter than both of these.



It is necessary to use a slightly more complicated integral than in the previous case to reach the result. The probability that the needle crosses one of the sides in the grid is:

$$\frac{2l(a+b) - l^2}{\pi ab}.$$

If $a = b$, the probability that the needle does not cut any line is:

$$1-\frac{l(4a-l)}{\pi a^2};$$

the probability of cutting one line is:

$$\frac{2l(2a-l)}{\pi a^2},$$

and of cutting two is:

$$\frac{l^2}{\pi a^2}.$$

We can generalise the problem of the needle by modifying the grid and making it, for example, triangular. But this is something we leave to specialists.

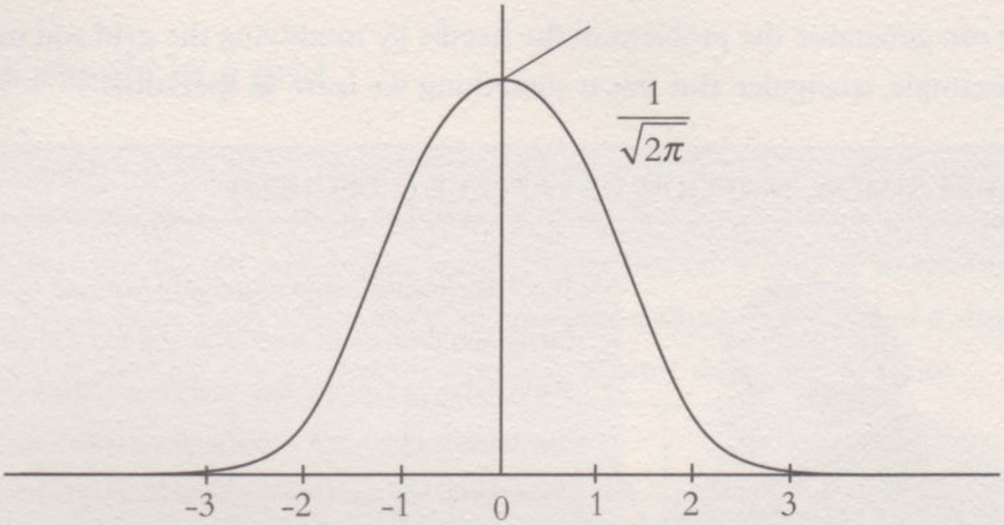
PIERRE-SIMON, MARQUIS DE LAPLACE (1749-1827)

This French astronomer and mathematician was a friend and protegé of Napoleon and the author of five volumes of *Celestial Mechanics* one of the fundamental texts of physics. Laplace, along with a precocious mind, had a prodigious ease for mathematical analysis and understanding of physical phenomena. He contributed to the development of many new concepts in probability (such as generating functions, conditional probability, Buffon's needle problem), pure mathematics (potential theory, Laplace transform, harmonic analysis) and astronomy (shape of the earth, primordial nebula, planetary instability) and can be almost universally considered a genius. His scientific achievements were so extraordinary to his contemporaries that, upon his death, his brain was severed from his body for in-depth study, but nothing new could be seen in him. He became one of Napoleon's ministers but this did not prevent him from accepting a noble title granted by the Bourbon Restoration. A famous anecdote tells how Napoleon, after being given a work on astronomy by Laplace, mentioned the conspicuous absence in the book of any mention of God. "Sir, I have not needed this hypothesis," came the answer.

The normal curve

In many issues related to probability and statistics, which could be the sharing or distribution of height, IQ, instrument errors of telescopes, the intensity of a laser, just to name a few examples, we encounter the Gaussian function 'bell' curve. It corresponds to a probability distribution that has a density curve in which π intervenes decisively.

The curve can be centred and standardised if we choose a zero mean and a variance of $\sigma^2 = 1$. Then the curve looks like a familiar bell, which we have exaggerated by drawing the ordinates larger



and which corresponds to the equation

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}.$$

The probability is calculated by means of an integral,

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-w^2}{2}}\, dw.$$

As can be seen, π is always present.

Normal distribution represents, for example, the age range in the case of deaths. We could say, to paraphrase Alexander Pope, that every time someone dies, the Gaussian bells toll for π.

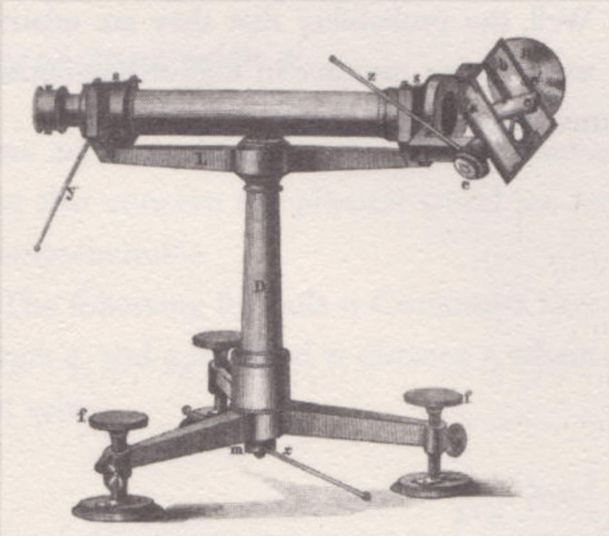
JOHANN CARL FRIEDRICH GAUSS (1777-1855)

No description can accurately reflect a monumental personality like that of mathematician, astronomer and physicist Gauss. Suffice to say that his contemporaries gave him the nickname *Princeps Mathematicorum* (Prince of Mathematics). Gauss, born into poverty, was a child prodigy in his home town. There is a story of his childhood, perhaps apocryphal but representative of his extraordinary talent nonetheless, which goes: A teacher proposed that the students add all the numbers between 1 and 100. In just a few minutes, Gauss came up with the correct answer: 5,050. How could the boy come up with a result that would have taken anyone else much longer? Gauss realised that there were 50 equivalents involved, all totalling 101:

$$1 + 100 = 2 + 99 = 3 + 98 = \ldots = 48 + 53 = 49 + 52 = 50 + 51.$$

Consequently, $50 \cdot 101 = 5{,}050$.

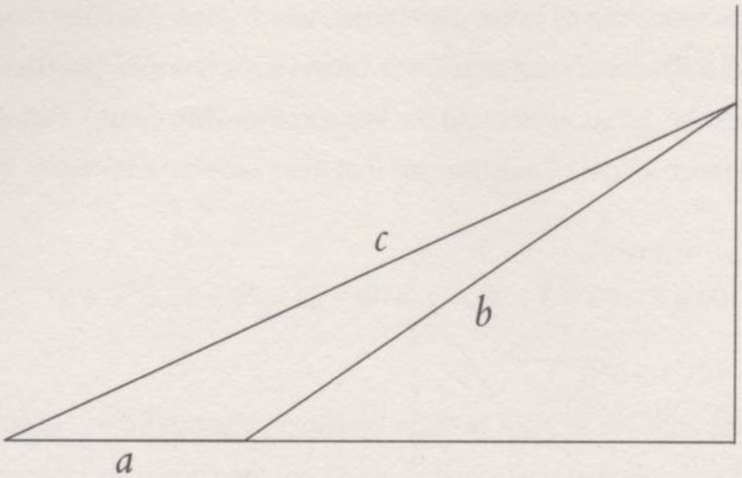
Gauss went on to be a prolific scientist because everything interested him. His contributions are numerous and staggering in their variety. He laid out the conditions for the constructibility of regular polygons, formulated the theorem for the distribution of prime numbers, showed the fundamental theorem of algebra, calculated the orbit of the dwarf planet Ceres, predicted essential points of non-Euclidean geometry, all in addition to obtaining many important results in analysis, algebra, number theory, probability theory and other branches of pure mathematics. In applied mathematics and physics, it is worth mentioning his work on surveying, electricity and magnetism, and the invention of the heliotrope, the heliograph and a version of the telegraph.



The heliotrope was one of the tools developed by Gauss for use in geodetic surveys. This device, which has a movable mirror, reflects sunlight in a predetermined direction, enabling precise alignments of topography instruments.

More probability with π

Here are some curiosities for the non-specialist but π-obsessed reader. If the measurements of a triangle with sides chosen at random are $a, b < 1$ y $c = 1$, the probability that a, b and c form an obtuse triangle is $\frac{\pi - 2}{4}$.



The average number of ways of writing (taking into account the order) a natural number as a sum of two squares is $\pi/4$.

We say that two Gaussian integers are complex numbers when they are of the form $x + yi$, with x and y whole. Gaussian integers can be added, subtracted, multiplied or divided, always resulting in other Gaussian integers. They constitute what in algebra is called a 'body'; they are kind of super-number integers, and support each other to define divisibility, with their greatest common divisor, their least common multiple and their primes. Well, the probability that they are relatively prime is $6K/\pi^2$, where K is a very well known number in higher mathematics, called Catalan's constant. Among normal integers, the probability is $6/\pi^2$.

Chapter 4

Formulae with π

The gentleman that serves the oracle of Delos does not reveal nor hide, but provides symbols.

Heraclitus

Lord Kelvin once wrote on the blackboard

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

and addressing his lecture audience proclaimed: "A mathematician is someone for whom this is as obvious as that 2 + 2 = 4."

We will not go as far as Lord Kelvin, but we will present formulae related to π. Writing formulae is an almost certain way of alienating the reader, so we have tried to minimise them and concentrate them into one chapter.

Some of them are almost mandatory knowledge for anyone interested in the subject and must be included. In other cases, although they may be difficult to understand, it is worth recognising the efforts and ingenuity it took to discover them.

Expressions with π

There are those formulae in which π intervenes and are always handy to know; those that concern the physical world are interesting and, in some cases, suitably incomprehensible.

The following formula is Coulomb's Law for the electrical force between two charges q_1 and q_2, situated at distance r, where ε_0 is the vacuum permittivity, a constant value:

$$F = \frac{|q_1 q_2|}{4\pi\varepsilon_0 r^2}.$$

This is Kepler's Third Law, with period *P,* masses m_1 and m_2, semi-major axis *a* and gravity constant *G*:

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)}a^3.$$

The Heisenberg Uncertainty Principle for a particle of average position *x* and average moment *p*, where *h* is a constant called Planck's quantum of action:

$$\Delta x \Delta p \geq \frac{h}{4\pi}.$$

Cosmological constant Λ, where *G* is the gravitational constant; *c*, the speed of light, and *p* is the density of matter and radiation:

$$\Lambda = \frac{8\pi G}{3c^2}p.$$

From there one can guess that what follows would only appeal to specialists, so we will not continue. It should be pointed out that these and other formulae from physics are not used to calculate π, but are useful to show some general knowledge.

Some mathematical formulae in which π is involved

The basic formulae mainly affect conical curves, so called because they result from cutting a cone with a plane. From now on, *r* will represent the radius:

Length of the circumference

$$L = 2\pi r.$$

Area of the circle:

$$A = \pi r^2.$$

Area of the ellipse of semi-axes *a* and *b:*

$$A = \pi ab.$$

Area of the regular polygon, with *n* sides and semiperimeter *s:*

$$A = \frac{1}{4} n s^2 \cot\left(\frac{\pi}{n}\right).$$

Area of the sphere:

$$A = 4\pi r^2.$$

Total area of revolution cylinder of height *h*:

$$A = 2\pi r(r + h).$$

Total area of the generatrix revolution cone *g*:

$$A = \pi r(r + g).$$

Volume of the sphere:

$$V = \frac{4}{3}\pi r^3.$$

Volume of ellipsoid of semi-axes *a*, *b* and *c*:

$$V = \frac{4}{3}\pi abc.$$

Volume of right cylinder of height *h*:

$$V = \pi r^2 h.$$

Volume of right cone of height *h*:

$$V = \frac{\pi r^2 h}{3}.$$

There are obviously other formulae in which π acts together with very complicated integrals.

Basic formulae

A basic formula is defined as any formula created prior to the advent of the computer age. After that date mathematicians turned their attention primarily to procedures likely to provide digits of π as effectively as possible; criteria such as mathematical beauty were set aside for the benefit of calculation. To quote formula after formula can be laborious, but there is probably no other choice:

$$\int_{-1}^{1}\sqrt{1-x^2}\,dx=\frac{\pi}{2}.$$

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}=\pi.$$

$$\int_{-\infty}\frac{dx}{1-x^2}=\pi.$$

$$\int_{0}^{1}\frac{x^4(1-x)^4}{1+x^2}\,dx=\frac{22}{7}-\pi.$$

$$\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt{\pi}.$$

$$\int_{0}^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

$$\oint\frac{dz}{z}=2\pi i.$$

The last integral is a complex and assumes that the integration path around $z = 0$ runs counter clockwise.

The series occupy an important place in the environment of π:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots=\operatorname{arc}\tan 1=\frac{\pi}{4}.$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots=\frac{\pi^2}{12}.$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots=\frac{\pi^3}{32}.$$

And they are of all types:

$$\frac{1}{4}(\pi-3)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k(2k+a)(2k+2)}=\frac{1}{2\cdot3\cdot4}-\frac{1}{4\cdot5\cdot6}+\frac{1}{6\cdot7\cdot8}-\ldots$$

$$\frac{\pi\sqrt{2}}{4}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\ldots$$

$$\frac{2^{24}76977927\pi^{26}}{27!}=\frac{1}{1^{26}}+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\ldots$$

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\operatorname{arc\,tan}\left(\frac{1}{n^2+n+1}\right).$$

$$\frac{11340}{691\pi^6}=\sum_{n\ prime}\frac{1}{n^6}=\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\frac{1}{13^6}+\ldots$$

And among them there are series linked to π and the mysterious Riemann zeta function, $\zeta(s)$:

$$\zeta(2)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}.$$

$$\zeta(4)=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots=\frac{\pi^4}{90}.$$

$$\zeta(6)=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\ldots=\frac{\pi^6}{945}.$$

$$\zeta(8)=1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\ldots=\frac{\pi^8}{9450}.$$

$$\zeta(2n)=\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\ldots=(-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}.$$

In the last case, B_{2n} are Bernoulli numbers, which are studied in superior mathematical analysis. Out of curiosity, we will look at the first ones:

n	0	1	2	4	6	8	10	12
B_n	1	–1/2	1/6	–1/30	1/42	–1/30	5/66	–691/2,730

Perhaps a simple list of series is not what the reader expects. We show how a simple example works, such as the first series of the list, called the Leibniz formula. If we start from the geometric series

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-\ldots,$$

that converges for $|x| < 1$, can be integrated part by part and make use of integral calculus to calculate

$$\operatorname{arc}\tan x=\operatorname{arc}\tan x-0=\operatorname{arc}\tan x-\operatorname{arc}\tan 0=\int_0^x\frac{1}{1+x^2}=$$
$$+\int_0^x(1-x^2+x^4-x^6+x^8-\ldots)\,dx=\int_0^x dx-\int_0^x x^2\,dx+\int_0^x x^4\,dx-$$
$$-\int_0^x x^6\,dx+\int_0^x x^4\,dx-\ldots=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\ldots$$

Now if we have $x = 1$, we see that we approach the end. To extend the validity of the calculation with $x = 1$, as we have already said it is only valid for $|x| < 1$, we must think a little more. Let's write the original formula of the geometric series but do it in a way that we dwell on the term n–1, then we write the rest as if it were the term n:

$$\frac{1}{1+x^2}=1-x^2+x^4-\ldots+(-1)^n x^{2n}+\frac{(-1)^{n+1}x^{2n+2}}{1+x^2}.$$

By integrating piece by piece *0* and *1* and giving x the value *1*, it leaves

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^n}{2n+1}+(-1)^{n+1}\int_0^1\frac{x^{2n+2}}{1+x^2}dx.$$

In this expression, upon taking the limit $\lim\limits_{n\to\infty}$, the last term tends towards zero. Therefore,

$$\arctan 1=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots$$

Unfortunately, the series is not very useful for calculating π, since its convergence is desperately slow. To get ten digits you must add together no less than 10^{50} terms; an astronomical amount. Of course, we will not repeat the process to show the why for each series. It would be long-winded and tiring, and would not reveal anything new to us. Some call this formula the Gregory-Leibniz series; in fact, it should be called the Madhava de Sangamagrama series, because it was this Indian mathematician who first formulated it. A curious series is:

$$\frac{\pi}{4}=\sum_{n=1}^{\infty}\arctan\frac{1}{F_{2n+1}},$$

in which the F_k are the Fibonacci numbers, components of the Fibonacci sequence, where each term is the sum of the previous two:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

We can also find π in the 'infinite products'. The following is due to John Wallis (1616-1703):

$$\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\ldots=\frac{\pi}{2},$$

and can be deduced from the integral

$$\int_0^{\pi}\sin x^n\,dx$$

after laborious algebraic manipulations. Lord Brouncker (1620-1684) converted the formula into a continuous fraction.

The following infinite product is down to Euler:

$$\prod_{n\ prime}\frac{n^2}{n^2-1}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\ldots=\frac{\pi^2}{6}.$$

It has the peculiarity of involving only powers of prime numbers. And this is another infinite product, even rarer,

$$\pi=\frac{2}{\prod_{k=1}^{\infty}\left(1+\frac{\operatorname{sen}\frac{1}{2}\pi p_k}{p_k}\right)},$$

that does not conform to involve prime numbers p_k, but is in a trigonometric format. There were no limits to Euler's imagination and ability to search.

LEONARDO OF PISA (1170 - 1250)

This Italian mathematician is known by many names, but after his death he began to be called Fibonacci (the son of Bonacci) and that's how we remember him now. His father was a merchant who fequently travelled abroad, and Fibonacci went with him, giving him a cosmopolitan education and a good knowledge of the Indian numbering system and Arabic mathematics. He became a highly respected figure in his time, a friend of the Holy Roman Emperor Frederick II, and even received a salary from the city of Pisa, who paid him to do little more than think. His focus was on practical problems like those to do with book-keeping and the mathematics of trade and so it was not always obvious how his work related to more abstract themes. Some of Fibonacci's results did not arouse interest until 300 years of his death. His best known work is the *Liber Abaci* of 1202, where we find the famous number series that carries his name,

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89…,

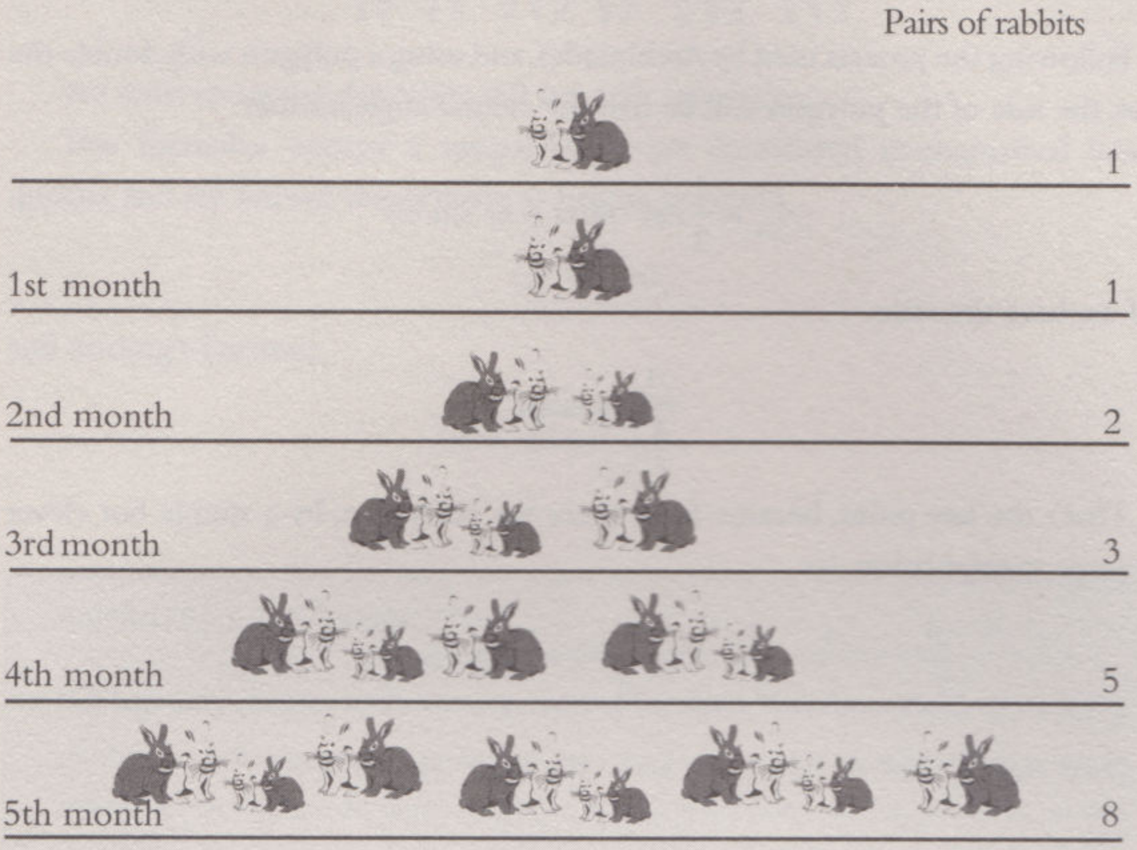
where each term is the sum of the previous two. Fibonacci applied it to a puzzle concerning families of rabbits. *Liber Abaci* contains other problems of the same type, but he became famous in Europe because he dealt with practical issues in easy-to-understand language. In other words, people did not seem to mind that he talked about π, but what really interested them is that he gave advice for the positional counting system using the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0, and that he spoke to them clearly about accounting, weights, measures and coins, benefit sharing, etc. Fibonacci wrote other books, dedicated to pure mathematics, relevant to geometry, equations and number theory, but they did not receive the acclaim they deserved at the time.

The classic result given below is due to François Viète (1540-1603), a French mathematician who in his time was called Vieta, the Latin form of his name:

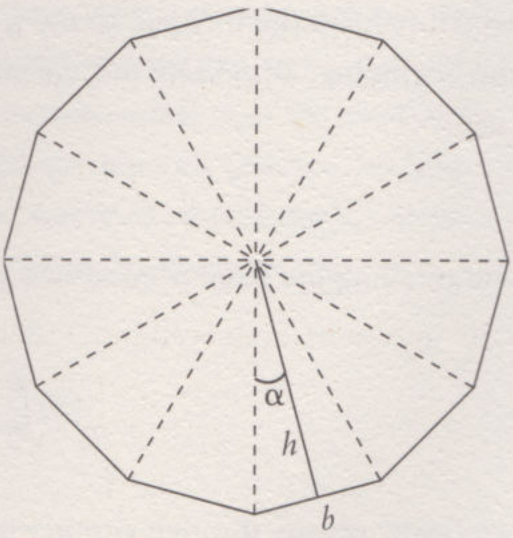
$$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\ldots=\frac{2}{\pi}.$$

This product, mathematically beautiful in itself, can be transformed algebraically into something even more sublime, as Joakim Munkhammar did in 2000:

$$\lim_{n\to\infty} 2^{n+1}\sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n}}=\pi.$$



The number of pairs of rabbits F_n in generation n, assuming there are no deaths and that it begins with a pair that does not procreate in the first month, obeys the law $F_n = F_{n-1} + F_{n-2}$. The F_n are the so called 'Fibonacci numbers'.



Viète's formula concludes, using modern language, as follows: starting with the triangle Archimedes used, but giving it a name, say α, the angle of the base of the triangle b determined by the height h and the side of the triangle in the figure.

We have, therefore, $A_n =$ *n(area of triangle)* and, using the tools of elementary trigonometry (which, incidentally, Viète himself helped to develop):

$$A_n = \frac{1}{2} nr^2 \sin 2\alpha = nr^2 \sin\ \alpha \cos\alpha.$$

Following the process used by Archimedes, and using a polygon with double the sides, the side of the polygon will be half the central angle, so that

$$A_{2n} = \frac{1}{2} 2nr^2 \sin\alpha = nr^2 \sin\ \alpha,$$

and we have the ratio

$$\frac{A_n}{A_{2n}} = \cos\alpha.$$

That's the key point, because from there we easily pass, by a simple but clever algebraic manipulation, to:

$$\frac{A_n}{A_{2^k n}} = \frac{A_n}{A_{2n}} \cdot \frac{A_{2n}}{A_{2^2 n}} \cdot \ldots \cdot \frac{A_{2^{k-1} n}}{A_{2^k n}} = \cos\alpha \cos\frac{\alpha}{2} \cdot \ldots \cdot \cos\frac{\alpha}{2^k}.$$

Now simply observe that, if we go to the limit when $k \to \infty$,

$$\lim_{k\to\infty} A_{2^k n} = \pi r^2.$$

And after another basic manipulation,

$$\neq = \frac{\frac{1}{2} n \sin 2\alpha}{\cos\alpha \cos\frac{\alpha}{2} \cos\frac{\alpha}{2^2} \cos\frac{\alpha}{2^3} \ldots}$$

which reduces the formula to the starting conditions. How much was α worth at the beginning? If, as Viète did, we take a square at the beginning,

$$n = 4, \cos\alpha = \sqrt{\frac{1}{2}},$$

and recalling the basic trigonometric double angle rule that says

$$\cos\frac{\chi}{2} = \sqrt{\frac{1}{2} + \frac{1}{2}\cos\chi},$$

we finally obtain, through another method, the desired expression:

$$\pi = \frac{2}{\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \ldots}$$

We must recognise that Viète did not lack ingenuity.

Two formulae occupy a special place, are considered mathematical beauty queens, and are known respectively as Euler's Formula,

$$e^{\pi i} + 1 = 0,$$

and Stirling's Formula,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

RIEMANN'S ZETA FUNCTION

This legendary complex function, designated by the Greek letter zeta (ζ) and the study of which might someday provide us with an unthinkable understanding about prime numbers and reveal many of their mysteries, is defined following the study of Euler, as a series, as well as an infinite product:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1 - p^{-s}}.$$

The function is defined naturally in the complex region consisting of the real component of complex numbers greater than 1, extends in a unique manner as Georg Friedrich Bernhard Riemann showed (1826-1866) for the entire complex plane in an analytic function and meromorphic with a single pole $s = 1$. The Riemann hypothesis states that non-trivial zeros of ζ are in the line formed by the real part of complex numbers equal to ½. All this seems (and is) quite difficult to grasp, and so is finding the link between π and the ζ function. The latter requires exploring the values of ζ(s), because π appears in all values corresponding to *s* integer and even. The next series is a curious one, found by Philippe Flajolet and Ilan Vardi:

$$\pi = \sum_{n=1}^{\infty} \frac{3^n - 1}{4^n} \zeta(n+1).$$

Turning to the continued fractions, the ways to calculate π do not become any easier to remember (it was Lambert who first calculated it):

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{}}}}}}}}}$$

There exist other, less canonical fractions, which are much more symmetric:

$$\pi = \cfrac{4}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \cfrac{25}{11 + \cfrac{36}{13 + \cfrac{49}{}}}}}}}}$$

$$\pi = 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + \cfrac{81}{6 + \cfrac{121}{}}}}}}$$

Lord Brouncker's formula is the last on the list, although algebraically manipulated. The beauty of continued fractions is that, when cut at a point in the 'tower', the resulting fraction is the best possible rational approximation of the number they represent. If we stop at a point in the development of the continued fraction of π and unravel the yarn the other way, we get the best possible rational approximation.

WHAT IS A CONTINUOUS FRACTION

Learning how to build a continuous fraction can be a complex task, but it provides a useful understanding.

Suppose that we work with a non-whole number N, which in principle would have a point and decimal after it. If we separate the integer part, which we call $[N]$, we get $N - [N]$, which, clearly, is between 0 and 1.

Now if we find the inverse of $N - [N]$, which $\frac{1}{N-[N]}$, is greater than 1 and we will simply call N_1, we will get:

$$N-[N]=\frac{1}{N_1} \quad \text{o} \quad N=[N]+\frac{1}{N_1}.$$

By separating the integer part of N_1 and repeating the procedure we will have a second fraction:

$$[N]+\cfrac{1}{[N_1]+\cfrac{1}{N_2}}.$$

And so on:

$$[N]+\cfrac{1}{[N_1]+\cfrac{1}{[N_2]+\cfrac{1}{N_3}}}.$$

If the process never ends, we get a tower of fractions:

$$[N]+\cfrac{1}{[N_1]+\cfrac{1}{[N_2]+\cfrac{1}{[N_3]+\cfrac{1}{\cdots}}}}.$$

When the process ends, N is revealed as a rational number, whole or fraction, or in decimals, finite or periodic. In the case of π, which is irrational, the continuous fraction is an infinite tower. The sequence, usually written

$$[[N]; [N_1], [N_2], [N_3]\ldots],$$

identifies N and the continuous fraction that represents it.

For example, if we stopped at [3, 7, 15] we would get

$$3+\cfrac{1}{7+\cfrac{1}{15}}=3+\cfrac{1}{\cfrac{106}{15}}=3+\frac{15}{106}=\frac{333}{106}\approx 3.141509\ldots$$

The fraction $\frac{333}{106}$ is the best rational approximation in the sense that anyone who improves it is bound to use a larger denominator. And the truth is that the approximation $\frac{333}{106}$ was found by Rivard in his day, and is an excellent approximation.

The so-called John Machin formula,

$$\frac{\pi}{4}=4\operatorname{arc\,tan}\frac{1}{5}-\operatorname{arc\,tan}\frac{1}{239},$$

and others have been derived and used to calculate the digits of π. Later we will see two of them, as Kanada used them to calculate 1,241,100,000,000 digits of π.

JOHN MACHIN (1680-1751)

The English mathematician served for 29 years as secretary of the Royal Society but has actually made the history books thanks to a single formula that bears his name. This, combined with the Taylor series, allows the calculation of π since its convergence is quite fast. Today many Machin-like formulas are known, such as

$$\begin{aligned}\frac{\pi}{4}=&183\operatorname{arctan}\frac{1}{239}+32\operatorname{arctan}\frac{1}{1023}-68\operatorname{arctan}\frac{1}{5832}+\\&+12\operatorname{arctan}\frac{1}{113021}-100\operatorname{arctan}\frac{1}{6826318}-12\operatorname{arctan}\frac{1}{33366019650}+\\&+12\operatorname{arctan}\frac{1}{43599522992503626068},\end{aligned}$$

due to Hwang Chien-Lih, who announced it in 2003.

Advanced formulae

The Indian mathematician Ramanujan paved the way when, around 1910, he unveiled the formula

$$\frac{1}{\pi}=\frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

(and 16 more), which has the remarkable property of producing eight decimal places each time it is calculated with one more term. But it took three quarters of a century to prove, as Ramanujan did not bother to do so. Bill Gosper, one of the first hackers in history, used it to calculate seventeen million digits. The variant

$$\frac{1}{\pi}=12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!(13591406+545140134n)}{(n!)^3(3n)!(640320^3)^{n+1/2}}$$

brought the figure from eight to fourteen decimal places. Besides that, the calculation could be divided between more than one machine.

The above formula, the work of the Chudnovsky brothers, saw the light of day in 1987, and we bring it up now to give an idea of how fast all aspects of computing move: in the 21st century the formula is used for PC calculations, not for advanced calculations.

These algebraic formulae may seem complicated, but that has not stopped them from appearing in a Walt Disney film, *High School Musical*: two of these formulae are written on the board and one of them even contains an error which is corrected in the action, highlighting the humour of the writers.

In 1946, with the ENIAC (Electronic Numerical Integrator And Computer), computers were introduced into calculus and everything changed, and, of course, so did the calculation of the digits of π. The ENIAC was the first functional electronic computer dedicated to pure computing applications. Its closest predecessor is the Colossus machine, used by Alan Turing (1912-1954) at Bletchley Park for military purposes, specifically to decode secret German messages. The ENIAC was built by John Presper Eckert (1919-1995) and John William Mauchly (1907-1980). Its size and power consumption have become legendary: it had almost 100,000 components

SRINIVASA RAMANUJAN (1887-1920)

This Indian was one of the most amazing mathematicians ever to live. Born into a poor family, he was self taught after reading only a mathematical summary without proofs or full descriptions. He wrote to several renowned European mathematicians, to whom he sent some of his results (120 theorems), but only received a response from the Cambridge don, G.H. Hardy (1877-1947), who read the manuscript one night in the presence of his colleague J.E. Littlewood (1885-1977) and could hardly believe his eyes. The formulae in the letter, as Hardy himself explained, must be true because "nobody would have enough imagination to invent them." In any case, they were amazing. Some were similar to results obtained by Hardy and Littlewood, and others were as original as they were strange. Later, first at Hardy's personal expense and then with a scholarship from the University of Cambridge, Ramanujan moved to Britain and was active there until his death, which came early due to tuberculosis. Despite the originality of his results, his mathematical contribution is as perplexing as it is difficult to assess. On many occasions he did not show his workings on the way to a formula.

As a person he was a very religious man and a strict vegetarian. His most popular story, known as 'the taxi story', portrays him well. Ramanujan was hospitalised due to the TB, and Hardy paid him a visit. He said he had come by taxi, specifically in 1,729, hoping, said Hardy, that it was not an unlucky number. "Not at all," was the answer. "It is the smallest possible natural number that is twice the sum of two different cubes." And it was true, because $1,729 = 9^3 + 10^3 = 1^3 + 12^3$, and 1,729 is the smallest number possible to prove that property. The story would lack importance if proving had not taken Hardy several weeks, and proper study of the problem took him nearly thirty-five years more. Today, we continue to study this kind of number; they are called 'taxicab numbers'.

including resistors, relays, diodes, vacuum tubes, capacitors, etc. It weighed over 27 tons, occupied a length of more than 30 m, and the heat it gave off raised the room temperature to 50°C. All that kit allowed it to do about 5,000 sums in one second, a thousand times faster than existing devices (and many millions times slower than a current PC). It could also store 200 digits in its memory. It was prgrammed by physically wiring togther components – taking several days to complete. It was a huge device, since there were no miniature transistors or circuitry. Nor was it complying with the Von Neumann architecture, where data and programs are held within the same memory.

The ENIAC took 70 hours to calculate the first 2,037 digits of π. The following table, which includes the year and the number of digits obtained, gives an idea of the significance of computing:

1949	D.F. Ferguson and John W. Wrench, using a non-electronic calculator	1,120
1949	John W. Wrench, Jr., and L.B. Smith already using the ENIAC	2,037
1958	François Genuys	10,000
1961	Daniel Shanks and John W. Wrench	100,265
1973	Jean Gilloud and Martin Bouyer	1,001,250
1983	Yasumasa Kanada and Yasunori Ushiro	10,013,395
1987	Yasumasa Kanada, Yoshiaki Tamura and Yoshinobu Kubo	134,214,700
1989	Gregory V. Chudnovsky and David V. Chudnovsky	1,011,196,691
2002	Yasumasa Kanada and a team of nine specialists	1,241,100,000,000
2009	Daisuke Takahashi and computer associates	2,576,980,370,000
2010	Fabrice Bellard	2,699,999,989,951

In 1973, an old Euler formula combined with another Machin-type formula allowed Gilloud and Boucalcular to find a million digits of π:

$$\pi = 864\sum_{n=0}^{\infty}\frac{(n!)^2 4^n}{(2n+1)!325^{n+1}} + 1824\sum_{n=0}^{\infty}\frac{(n!)^2 4^n}{(2n+1)!3250^{n+1}} - 20 \operatorname{arc}\tan\frac{1}{239}.$$

The funny thing is that the second sum is enough to calculate the first albeit at a different decimal place. However large they are, both sums differ only by the number of zeros in front of them.

In 1976 the Eugène Salamin-Richard Brent algorithm made its appearance, based on an old idea of Gauss and Legendre – iterations of arithmetic-geometric media. Although difficult to explain in a nutshell, we say that an algebraic algorithm is just a recipe to calculate something, in this case π. Salamin and Brent started from the definitions

$$a_0 = 1; \quad b_0 = \frac{1}{\sqrt{2}}; \quad t_0 = \frac{1}{4}; \quad p_0 = 1$$

and then recurrently computed

$$a_{n+1} = \frac{a_n + b_n}{2};$$

$$b_{n+1} = \sqrt{a_n b_n};$$

$$t_{n+1} = t_n - p_n(a_n - a_n + 1)^2;$$

$$p_{n+1} = 2p_n.$$

In the limit it is verified that

$$\pi \approx \frac{(a_n + b_n)^2}{4t_n}.$$

It is unthinkable to calculate the above algorithm with pencil and paper or even using mechanical and electric calculators because of its quadratic convergence, meaning that it increases in complexity as it requires the doubling of the figures obtained in the previous step. However, using a computer the first 206,158,430,000 digits of π were calculated with this algorithm.

But that's not all. In the 1980s, Peter and Jonathan Borwein created a fourth-degree algorithm, which will not be reproduced here, since it is purely a field for specialists, that would allow π to approach 1,241,100,000,000 digits.

THE MAGNIFICENT FOUR

Any π specialist knows who the Borweins are, an exceptional Canadian family. The father, David Borwein (b. 1924), from Lithuania, is a mathematical icon in his adopted country. He has worked in many fields; mostly in series. The oldest son, Jonathan (b. 1951), a prolific author, is a leading figure in computing with a special appreciation for the digits of the number π. He loves teaching mathematics and makes specific software on the subject. Peter (b. 1953) is one of the creators of the BBP formula (named in honour of its discoverers, Bailey, Borwein and Plouffe) to calculate π. He is another computer genius. The mother of Jonathan and Peter, wife of David, is also distinguished scientist, not in mathematics but in anatomy.

In late 2002 a Japanese team led by specialist Yasumasa Kanada moved the milestone yet again. And that was not the the end of the story. Progress in this field seems unstoppable, and in 2010 π reached 2,699,999,989,951 digits.

Looking back to 1983, the expert Daniel Shanks (not be confused with William Shanks) thought that calculating a billion figures of π would be an insurmountable task... As a testimony to history, we reproduce here the two self-correcting (Machin type) formulae used by Kanada:

$$\frac{\pi}{4} = 12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443};$$

$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943}.$$

The first is from 1982, but the second, a creation of F. C. W. Störmer, is from as long ago as 1896 (published in French in the Bulletin de la Société Mathématique de France). Who would have said anything to Störmer! In mathematics you never know: what seems irrelevant today may be critical tomorrow.

More than the amazing records, the out-of-the-ordinary computation methods may call our attention. Use of the formula

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

made it possible to calculate any figure n of π without needing to compute all of the preceding digits of n. Of course, unfortunately, such a feat only gives binary or hexadecimal numbers. Formulae like the above are due to David H. Bailey, Peter Borwein and Simon Plouffe, and are known as BBP formulae, from the initials of their surnames. It is assumed they will mark the beginning of a new era of computation.

Fabrice Bellard's (b. 1972) formula,

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(\frac{-2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right),$$

a derivation of the BBP formula, is already 43% faster than the original.

Base 2 calculation uses simple bits (0 or 1) and has already surpassed one trillion digits. We know that in the trillionth place the figure is 0 (there are only two possibilities, 0 and 1), although, as dictated by the algorithm, we do not know the digits in between. The knowledge does not take up space, although its actual usefulness is questionable.

In conclusion, we can say that formulae already exist that allow one to find a digit in the development of π in any base.

ONE DIGIT WITHOUT A FORMULA

The columnist from the *Scientific American* Martin Gardner (b. 1914), a noted writer, magician, debater and mathematician, predicted in 1966 that the one millionth decimal place in the development of π would be a 5. It was based on an authorised English version of the Bible and specifically in Book 3, Chapter 14, verse 16 (3.1416), which depicts the magical number 7, and the seventh word has 5 letters. So the one millionth digit of π (then unknown) should be a 5. No one put much faith in this prediction, but in 1974 the digit was found, and it was a 5. Martin Gardner did not use any formula at all.

Beyond formulae

We finish the chapter with some interesting mathematics that can be considered formulae. For example,

$$i^i = e^{-\pi/2},$$

an equality that connects the real and complex fields.

The following equality links π with the world of prime numbers:

$$\frac{3}{\pi^2} = \lim_{n\to\infty} \frac{\Phi(n)}{n^2},$$

where

$$\Phi(n) = \sum_{k=1}^{n} \phi(k).$$

The $\phi(k)$ are the number of integers lower than k and primes with k.

The following formula is found in a chapter on number theory within the quasi-whole numbers. Evidently they are irrational, but when calculated with a conventional calculator appear to be integers. The difference between them and a truly whole number resides in a hidden decimal digit and requires a very good calculator to see that there is something significant hidden behind a long string of zeros after the decimal point. Below is an example of number that is almost 427 but not quite and π is involved:

$$427 \approx \left(\frac{\log(5280^3)(236674 + 30303\sqrt{61})^3 + 744)}{\pi} \right)^2.$$

If you calculate with any regular machine, the complex square gives 427, calculated more carefully, using a better calculator, the difference between the integer 427 appears at the 52nd decimal place. At this digit, the zeroes stop, and the calculator seems to come back to life. This square really is best described as quasi-whole.

It cannot be denied that the tentacles of π cover many fields. As an example, let's use one as far away as the Kepler conjecture on packing. What is the maximum number of disks that can be packed onto a plane? The answer is:

$$\frac{\pi}{2\sqrt{3}}.$$

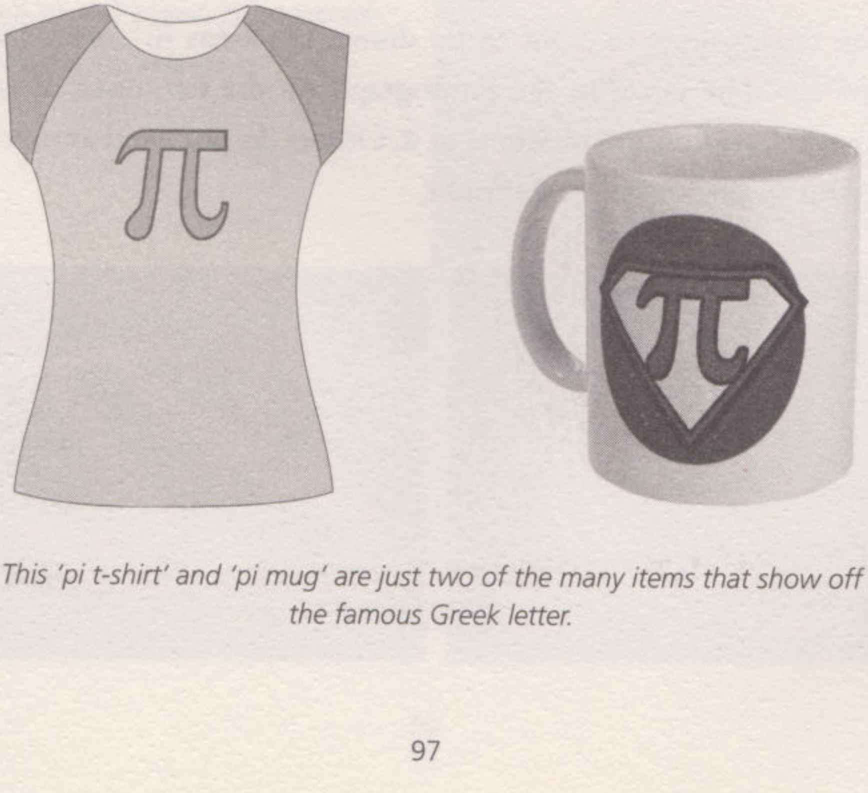
Chapter 5

Pi-mania

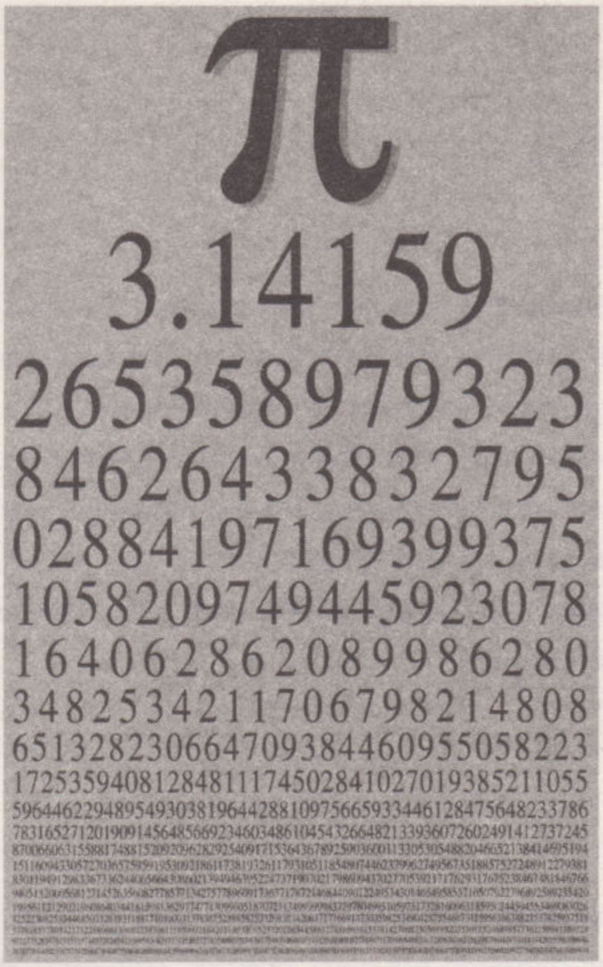
The number π is something special. We have said that it is the best known and most mentioned of all the numbers; the most famous. The enthusiasm, passion, true fanaticism for π and its digits have led to coining of the term 'pi-mania' which is described as an obsession with everything about this number. A walk through this world, halfway between the extravagant and the academic, is entertaining as well as interesting. It may seem to the reader that we are a bit over thetop in devoting so much space to a subject that has little to do with maths, but the reality is that π is really more than just a number.

Orbiting π

Even the world of marketing revolves around π. It is possible to find T-shirts with ideograms of π (even designed for pets), buttons, cuff-links, mugs, teapots, clocks, mouse pads, aprons, teddy bears, pillows, boxes, tiles, hats, posters, car trims and much more.

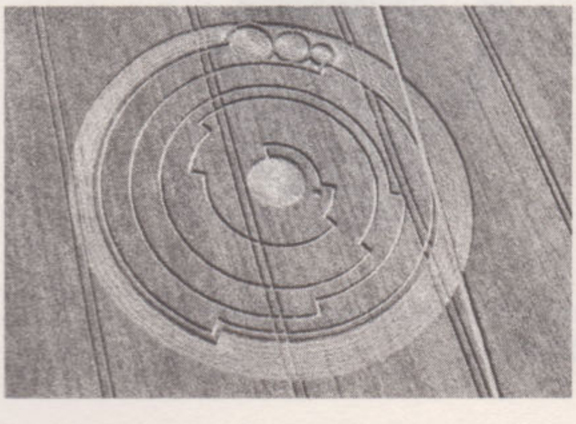


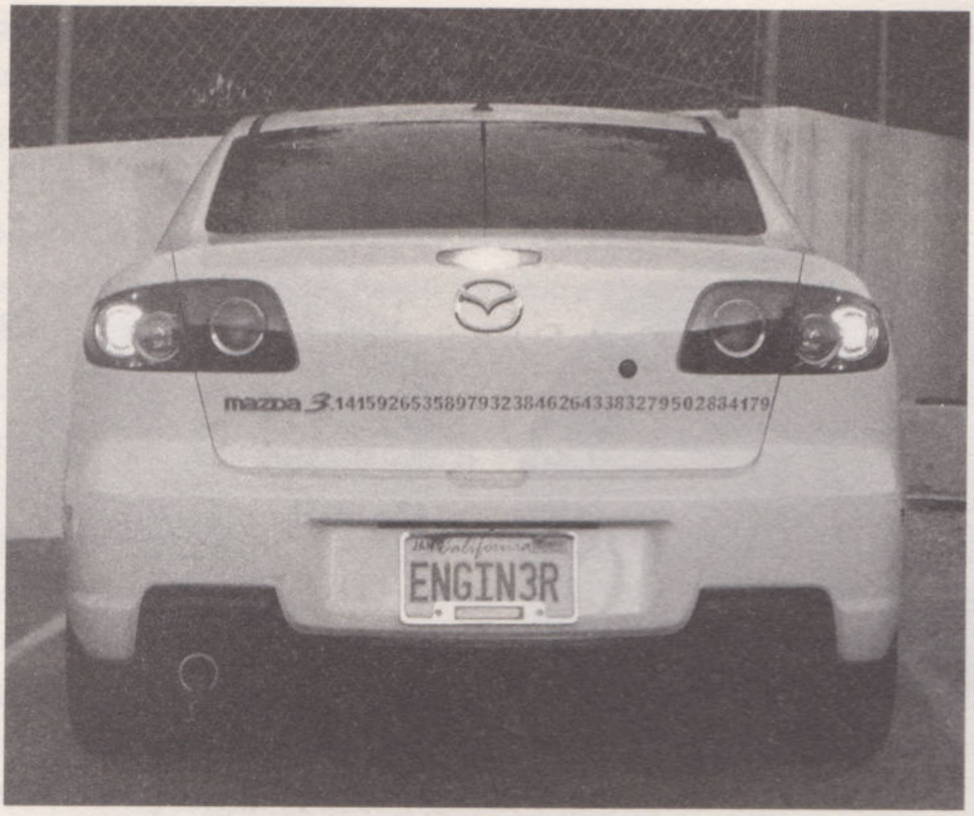
This 'pi t-shirt' and 'pi mug' are just two of the many items that show off the famous Greek letter.



An optician's poster for fans of the number π.

The crop circles once thought to be drawn by aliens in cornfields had their heyday years ago. The circle in the photograph on the left, once unravelled, was discovered to be based on the numbers of π. Others do not need as much detective work, as shown in the image on the right.

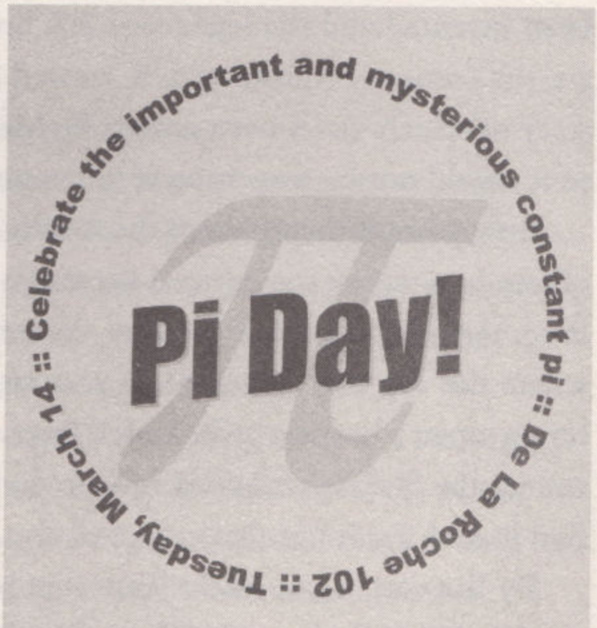




This Mazda 3 has been converted to a Mazda π.

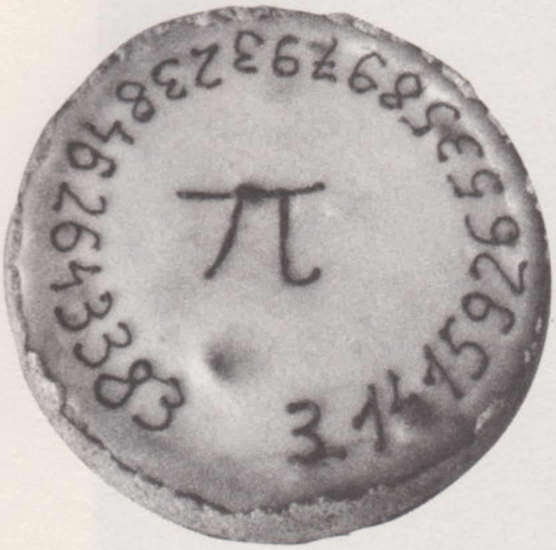
Pi Day is 14 March, claims its creator, American Larry Shaw, since 14 March is written 3/14 or 3-14 in American notation and those are the first few digits of π. This may seem absurd, but it has had great success.

The traditional meal at Pi Day celebrations is a round pizza, encompassing as it does the secrets of π.


Celebrate the important and mysterious constant pi :: De La Roche 102 :: Tuesday, March 14 ::
Pi Day!
π

A poster showcasing the celebration of Pi Day, which is held on 14 March every year.

Pi Day also coincides with the birthday of Albert Einstein, which undoubtedly contributes to the success of the celebration. It is possible to find the number π in other food products, like cheese or wine, and even perfume.



A pizza 'pi' to celebrate Pi Day can be served on 'pi plates' like this.

The symbol π has a reputation that extends across time. In 1915 it was the emblem of the RAF's 22 squadron. When Google issued shares, Series A numbered shares reached 14,159,265 (remember that π = 3.**14159265**…). The mathematical background of the owners of Google was there to see on the trading floors of Wall Street.

In 1982, in the heroic days of computer gaming, when consoles still had not been invented and the legendary ZX Spectrum or Dragon 32 were state of the art, the company Automata UK created a game entitled *Pimania,* the main character of which was a hero named Pi-Man. Currently the game is very outdated, so it would not be surprising if someone decides to relaunch it.

Perhaps most surprising of the π cults is a search engine that finds any sequence of numbers within the decimal expression of π. Up against the likes of Google and Bing, this search engine has zero commercial value. If someone wants to know where the day, month and birth year appears in the infinite decimal places of π, simply open a browser at Pi Search Page and write in the number and if it is there among the first two hundred million digits a message will tell you where you can find it. And if it is not there you will also be told.

By the way, if you enter your abbreviated date of birth (eg 18/11/46 is entered as 181146), the probability of getting an answer is almost 100%; if searching for only four digits (say, born 1/3/56, or 1 March 1956), a hit is assured because by decimal place 60,872 of π all of the possible combinations of four digits have already appeared. If, however, one prefers a more complete date (18/11/1946, for example, entered as 18111946), the probability of finding oneself drops to 63%. The longer the sequence of numbers, the less likely a hit will be.

A variant of the previous hobby is locating a phone number, car registration or any other number among the decimals of π; it offers no new developments regarding the search for birth dates.

Using a super-fast search engine to work in the decimal expansion of π reveals curious loops. For example, results have been found with a loop starting with the digits '40', which appear in a position in the seventies, followed by '70' and so on, starting a seemingly endless series. But in fact it does have an end. Dan Sikorski found the following sequence: 40, 70, 96, 180, 3664, 24717, 15492, 84198, 65489, 3725, 16974, 41702, 3788, 5757, 1958, 14609, 62892, 44745, 9385,169, 40, which returned, after many detours, to its origin. A loop just waiting for a probability study about its appearance.

Poems and mnemonics

Almost every time you want to quote a poem about π the same source is used. Wieslawa Szymborska, a Polish poet, who in 1966 was awarded the Nobel Prize for Literature, wrote a superb piece called *The Great Number* which has many translations. One of them reads:

"The admirable number pi: three point one four one.
All the following digits are also first, five nine two because it never ends.
It can't be comprehended six five three five at a glance, eight nine by calculation, seven nine or imagination, not even three two three eight by wit, that is, by comparison four six to anything else two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi doesn't stop at the page's edge. It goes on across the table, through the air, over a wall, a leaf, a bird's nest, clouds, straight into the sky, through all the bottomless, bloated heavens.
Oh how brief – a mouse tail, a pigtail – is the tail of a comet! How feeble the

> star's ray, bent by bumping up against space!
>
> While here we have two three fifteen three hundred nineteen my phone number your shirt size the year nineteen hundred and seventy-three the sixth floor the number of inhabitants sixty-five cents hip measurement two fingers a charade, a code, in which we find hail to thee, blithe spirit, bird thou never were alongside ladies and gentlemen, no cause for alarm, as well as heaven and earth shall pass away, but not the number pi, oh no, nothing doing, it keeps right on with its rather remarkable five, its uncommonly fine eight, its far from final seven, nudging, always nudging a sluggish eternity to continue."

Many other "odes to π" and the like have been written. Many π fans think of poetry only as an instrument; thus was born Pi-philology, a mnemonic based on the use of poems to remember numbers of π. These compositions are called "piems". Normally, each word of a poem represents a figure of π, its length in letters. Thus, 'paradise' is equivalent to 8, it is an eight-letter word. "Honey" reads as 5; it has five letters. Saying "Honey, your eyes take me to paradise" is merely a code for the decimal numerical sequence 5444228, and not in honour of your lover's eyes.

HONEY, YOUR EYES TAKE ME TO PARADISE
5 4 4 4 2 2 8

This example shows a mnemonic technique for remembering numerical sequences that appear random or governed by chance, like the decimal places of π. Want to remember them? Then look for these phrases and poems and memorise them. Though it doesn't seem like it, it is always easier to remember verses than figures. Just bear in mind that every word is worth the number of letters that compose it; you just have to count them.

This reduces the value of poetry to a simple instrument, and a poem becomes valuable not for its beauty, but for its length, if it reminds us of a large number of figures. Here are some examples (it should be noted that some verge on the absurd and there are a multitude of languages).

"Pie
I wish I could determine pi
Eureka, cried the great inventor
Christmas pudding, Christmas pie
Is the problem's very centre."

It provides the sequence of 20 decimal places:

3.14159265358979323846

Another composition takes us to 30 digits:

"Sir, I send a rhyme excelling,
In sacred truth and rigid spelling,
Numerical sprites elucidate,
For me the lexicon's full weight,
If nature gain, not you complain
Tho' Dr Johnson fulminate."

3.141592653589793238462643383279

There are other more complicated encodings that allow one to reach almost to 80 decimal places. In 1986 one English piem encoded 400 figures. Other encodings have the advantage of brevity, with phrases that can be memorised easily:

"May I have a large container of coffee right now please?"

Here, the last digit (6) is rounded up, since it is 5 followed by an 8.

The most popular way to remember the first fourteen decimal is the phrase, attributed to the Nobel laureate Sir James Hopwood Jeans (1877-1946):

"How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics!"

3.14159265358979

This can be extended even further by adding a postscript:

"How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics and if the lectures were boring or tiring, then any odd thinking was on quartic equations again."

But it stops there on the 32nd word, apparently inexplicably. Many mnemonic sentences and verses necessarily stop at decimal 32, because it is a zero, and to express it in words becomes an almost insurmountable obstacle. In French we find the following example:

"Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, artiste ingénieur,
Qui de ton jugement peut priser la valeur ?
Pour moi, ton problème eut de pareils avantages.
Jadis, mystérieux, un problème bloquait
Tout l'admirable procédé, l'œuvre grandiose
Que Pythagore découvrit aux anciens Grecs.
Ô quadrature! Vieux tourment du philosophe
Insoluble rondeur, trop longtemps vous avez
Défié Pythagore et ses imitateurs.
Comment intégrer l'espace plan circulaire ?
Former un triangle auquel il équivaudra ?
Nouvelle invention : Archimède inscrira
Dedans un hexagone ; appréciera son aire
Fonction du rayon. Pas trop ne s'y tiendra :
Dédoublera chaque élément antérieur ;
Toujours de l'orbe calculée approchera ;
Définira limite ; enfin, l'arc, le limiteur
De cet inquiétant cercle, ennemi trop rebelle
Professeur, enseignez son problème avec zèle."

How I like to teach this useful number to the wise!
Immortal Archimedes, artist and engineer,
in your opinion who could estimate its value?

For me, your problem had equal advantages.
Long ago, a mysterious problem blocked
All the honorable process, the great work
That Pythagoras revealed to the Ancient Greeks.
Oh quadrature! Old philosopher's torment
Unsolvable roundness, for too long you have defied
Pythagoras and his imitators.
How to integrate the flat circular space?
Form a triangle to which it is equivalent?
New invention: Archimedes will inscribe
Inside a hexagon; he will calculate its area
As a function of a radius, it will hold on to very little,
It will split each previous element
Always approaching the calculated area
It will define the limit; finally, the limiting arc
Of this disturbing circle, an enemy too rebellious
Teacher, teach its problem with zeal.

There are 126 decimal places represented. Note that the 10 letters of "mystérieux" introduces the elusive 0. Similar tricks with words of more than nine letters are used to represent other occurrences of zero.

One last thing: in dealing with piems be sure to count. George Gamow, a famous scientist (among other achievements, he was one of the fathers of modern nuclear physics concepts applied to star formations) once was reproached in *Scientific American* that he had mentioned π as 3.14158 instead of the correct 3.14159. The reason is that Gamow, who was Russian, a nationalised American citizen and a polyglot, misspelled the verse *"Que j'aime à faire apprendre"* of a French version of a π mnemonic… and wrote it wrong, having dropped one of the ps of *apprendre*.

Haiku holds a special place in poetic forms, the special Japanese tercet, or three-lined verse. They are written in many languages and places, and π would not be an exception. Although there are haikus dealing with π, the resourcefulness lies in composing a *piku*, a mathematical variant of a haiku. While a haiku has three lines trio with the 5-7-5 syllabic structure (ie the first line has 5 syllables, the second 7, and the third, again 5), a piku, which is also a tercet, has one more requirement: its words should have a number of letters equal to the corresponding decimal of π.

We offer up a piku that is, also, self-referential, as it describes π:

Can I know a cycle,
according to nature round
and never complete?"

Other systems use existing poems and special coding systems. Among those engaged in such unusual activity is Mike Keith, an enthusiastic pi-maniac to whom we owe a unique version of the poem *The Raven* by Edgar Allan Poe. Keith's version is called *Near a Raven*. But that's nothing: Keith himself has increased his repertoire and has written *Cadaeic Cadenza,* a major work of substance that expands the previous poem with fragments of Lewis Carroll, Omar al-Khayyam, William Shakespeare and other authors. *Cadaeic Cadenza* stores 3,834 digits, which is quite an achievement. The title *Cadaeic* perhaps needs some explanation:

C a d a e i c
3. 1 4 1 5 9 3

A different method can be used to remember the numbers of π; it is an activity somewhere between sport and the mystical, indefinable but real. It is no wonder so many cultivators of this hobby, real memory athletes, appear in the pages of the *Guinness Book of Records*... and sometimes in maths books.

Memory records are a fast-changing subject, with each memoriser trying to beat the score set by a predecessor. There is even a division among the memory experts with some aiming for a high number of digits while others go for speed. But we will not go into subtleties. The Ukrainian Andriy Slyusarchuk, claims to have remembered thirty million numbers, although the *Guinness Book of Records* does not recognise this as a verifiable fact.

The accepted record was set in 2006 at a hundred thousand digits at the hand of Akira Haraguchi, Japanese. However, let's not think that only pi-maniacs are entertained in these tasks, because if we reach into the list of some true memory experts we find internationally recognised scientific luminaries such as American Alexander Aiken and Canadian Simon Plouffe.

Pi-music

In the field of music, π does not make many appearances, despite the strong links between maths and melody. Remember that the quasi-official system in the Western world is tuning with intervals of equal value $\sqrt[12]{2}$. The diatonic scale, much older, is defined more or less as what the experts call the 'fifth interval', to which the value 3/2 is attributed. All these techniques only serve as a prelude to what comes next.

Tuning

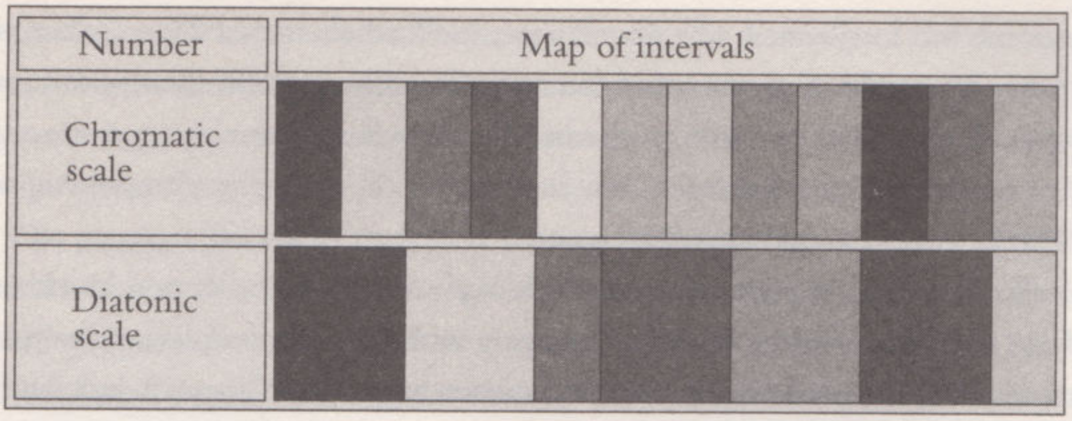
The Western musical scale has its origin in Pythagoras and in the notes of the diatonic scale called (C, D, E, F, G, A, B). The notes in the Pythagorean version correspond to the frequency of vibration of a string. Roughly speaking, at different frequencies, there are different notes. The separation of vibration (notes) was measured at intervals, which are not the result, as one might think, of subtracting one frequency from another, but in dividing them among themselves. The ratio between two frequencies gave simple and elegant fractions, such as the fifth interval, which indicated the frequency ratio from any one note and the note five steps away – 3/2. These intervals culminate in the octave, when the scale starts again on the same note, corresponding to the fraction 2/1= 2. For example: if you press the centre key of a piano, a note will sound, *A*, at a frequency of 440 Hz (wavelengths per second). Pressing the next *A*, located seven white keys to the right, the new sound will have a frequency of 880 Hz. The acoustic difference between the two notes (the interval between them) is expressed in music with the ratio 880/440 = 2.

This interval is called the 'octave'. Two notes separated by an interval of an octave (in our example, *A*) sound exactly the same, only with different wavelengths. Pythagoras, who had no piano, verified this by strumming a simple string, because if you pluck a string double the length of another, both plucks will provide the same notes.

The problem of this scale is that, although it worked well with instruments like the violin, it did not measure some small differences between notes, for example, those between sharps and flats. This led to the introduction of 'good character': the equation of sharps and flats to the rest of the notes by taking twelve equal intervals, each with amplitude $\sqrt[12]{2}$ with respect to the next. The resulting scale of twelve notes was baptised as the chromatic, or equal-tempered, scale.

The value $\sqrt[12]{2}$ results in the following problem: whether to divide an interval of size 2 into equal parts and how to establish the measure by which to divide the frequencies. To do this requires the creation of twelve consecutive intervals in a geometric progression, the radius of which is $\sqrt[12]{2}$. This series of twelve intervals, one for each note, is:

$$\sqrt[12]{2},(\sqrt[12]{2})^2,(\sqrt[12]{2})^3,(\sqrt[12]{2})^4,(\sqrt[12]{2})^5,(\sqrt[12]{2})^6,(\sqrt[12]{2})^7,(\sqrt[12]{2})^8,(\sqrt[12]{2})^9,(\sqrt[12]{2})^{10},(\sqrt[12]{2})^{11},(\sqrt[12]{2})^{12}=2.$$



The notes of the chromatic scale, from left to right: C, C#, D, D#, E, F, F#, G, G#, A, B♭, B, C. *The notes in the diatonic scale are:* C, D, E, F, G, A, B, C.

In the 18th century a new interval system was developed in which the fifth was worth $600 + 300/\pi$. Charles Lucy (b. 1946) has developed this new tuning and it bears his name.

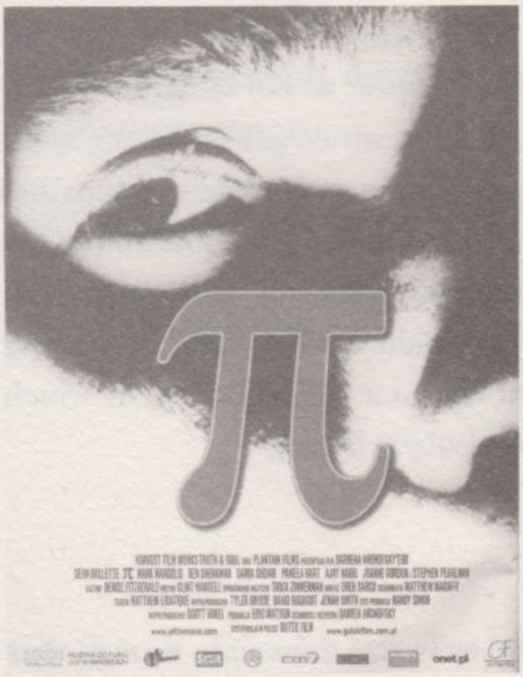
Another musical aspect of π, more fun than intellectual, is the possibility of 'hearing' the decimals of the number. There are programs on the Internet that 'play π'. Specifically, they assign a musical note to each of the ten digits within the constant. Each time a digit appears, the program assigns the corresponding note and plays it over the speakers. As the digits of π are (or appear to be) random, what is heard is a random composition. Most likely the listener would get bored, the more digits there are the less you may want to listen, but it's always possible that listening to a very, very long random sound could produce pleasure for someone. In the diatonic scale there are 7^{10} distinct ways of constructing a melody with seven pure notes (without sharps or flats) in combination we would say there are 7^{10} repeating combinations. From these possibilities we must discount the sequences in which the same note repeats, as the result would be particularly boring.

Cinema, literature and π

The appearances of π on the silver screen number in the thousands, though they are generally reduced to mere anecdotes that serve to add a halo of mystery to certain situations, such as in the case of the code name of the organisation of escapees in *Torn Courtain* by Alfred Hitchcock. The mathematical nature of π is rarely explored, and if it is, it is done in a gimmicky or superficial manner, such as in the case of π (or *Pi*), a 1998 film directed by American director Darren Aronofsky. The film focuses on the intellectual adventure of genius mathematician Max Cohen, immersed in a process of progressive insanity induced by his obsession with certain numbers. It plays with the idea of the hidden meaning of numbers, a notion akin to that of the Hebrew Kabbalah, but at all times the dramatic component takes precedence over any mathematical rigour.

A film in which π played a central role was in the science-fiction blockbuster *Contact* (1997), directed by Robert Zemeckis, with Jodie Foster playing the part of astronomer Eleanor Arroway. It could have been a great film about π, but the final script left the constant out in the cold. In the original novel, *Contact* (1985), from renowned cosmologist Carl Sagan, the book on which the feature film was based, π is extraordinarily important, to the point of raising a scientific-level debate. But one thing at a time.

Contact is a film about the possible contact with extraterrestrial civilisations and the problems, above all, religious problems, that such an encounter could create between us. It tells the adventure of Dr. Arroway, starting with her job as a researcher in charge of several radio telescopes, detecting a signal from outer space, the construction of a device to generate a 'wormhole' in the fabric of the universe and



Above, a poster for Pi, *a 'different' type of film. In 1997, Robert Zemeckis directed the film adaptation of the Carl Sagan novel* Contact, *in which alien beings allege there is a message hidden among the decimals of* π.

then make contact with aliens. They pose the idea to Arroway of the chance that there is a message hidden within the digits of π. A message hidden in the univerasl constant about the nature of reality is something that not even the aliens could manipulate, as this privilege is available only to God, creator of the universe.

The message, the aliens suggest, is written within the development of π. There is an enormous fragment of the digits of π, formed only by zeros and ones, that could be placed along the length of a square and form a perfect circle. It has been written there since all eternity in nature, within π, so, did God put it there? In the last chapter we will talk about *Contact* and the decimal digits of π, including the possibility of the existence of a just such a square and a circle of zeros.

In the modern classic of humorous literature, the *The Hitchhiker's Guide to the Galaxy* by Douglas Adams a giant computer communicates to man the answers to the meaning of life, the universe and everything. The answer, a little surprisingly, is '42'. For some pi-maniacs the computer's response was beyond a joke and was thought to be provocative, so they set to work. They tossed away 42 as too simple and searched threough the digits of π for somewhat more creative combinations, such as 424242. This showed up in position 242,423. It is almost certain that this rather ridiculous search will continue.

The number π and the law

It is a bit strange here to see a section on something seemingly so distant from π as law, legislation and related issues, but this number is in all areas. Since as far back as 1836 and in advanced post-revolutionary France there was a citizen-scientist, La-Comm, who not only claimed that π amounted to 3.25, but, surprisingly, received awards from several institutions for his discoveries about π. And this occurred when more than 100 digits of π had already been found.

Better known legislative news comes form Indiana, in the United States. There, in the year 1897, a citizen named Edward Goodwin who had shown his aptitude for geometric gymnastics, having even published something (only a headline and a short excerpt) in *American Mathematical Monthly*, persuaded the state legislators to approve the text of a proposed law (Bill 246) that was to be endorsed by the state House of Representatives.

Up until this point everything seems normal. What was weird was this law postulated, as deduced from many laborious calculations, that

$$\pi = \frac{16\sqrt{2}}{7} \approx 3.232.$$

It also sought that this equation should be incorporated into the state's textbooks; and Mr. Goodwin would be paid a royalty. In short, Goodwin and the eventual legislators did not take into account the fact, as shown by Lindemann more than thirty years before, that π was transcendental, the corresponding impossibility of squaring the circle and that through centuries of history more than one hundred decimal places of π had already been calculated. None of this worried the heroes of Indiana.

The bill was approved by more than one house committee and passed to the State Senate with the recommendation that it should be voted in. Fortunately, albeit randomly, this was completed. Someone showed the text to a professional who happened to be passing through, Professor Clarence Abiathar Waldo (1852-1926) so he could write a foreword to the text, and he replied politely that he was already a bit tired of people claiming to have squared the circle. Then, horrified to read the text in detail, he put senators right. Upon hearing from Waldo about how absurd the law was, finally, the legislators of the higher house did not accept the famous Bill 246.

Thanks to mathematics professor Clarence Abiathar Waldo, the Indiana state legislature avoided making a mathematical fool of itself.

The number π and art

In the Palais de la Découverte in Paris there is a frieze dedicated to π; it deals with an image that characterises the place. The frieze shows more than 600 decimals calculated by William Shanks in 1873. In fact, the eminent British mathematician got to 707, but D. F. Fergusson discovered in 1944 that he had made a mistake at digit 528. The following limerick from Nicholas Rose were created in his honour:

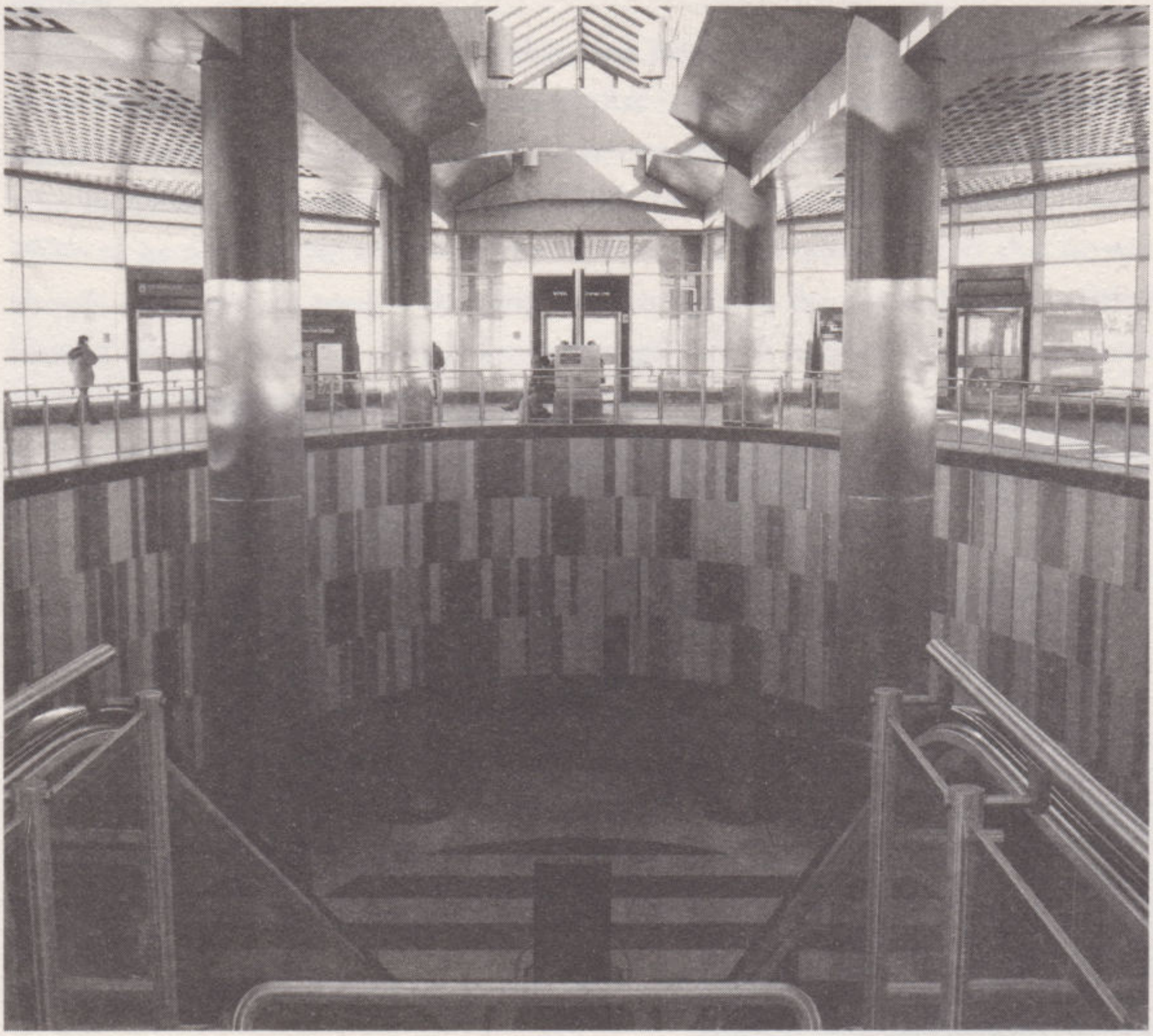
Seven hundred seven, Shanks did state,
Digits of π he would calculate.
And none can deny
It was a good try
But he had erred in five twenty eight."

π=3,141592653589793238462643
ABEL ABOUL-WEFA AHMES ALBATANI ALKHOWARIZMI AMPERE APOLLONIUS ARCHIMEDE

Frieze of the Palais de la Découverte showing the first 600 digits of π.

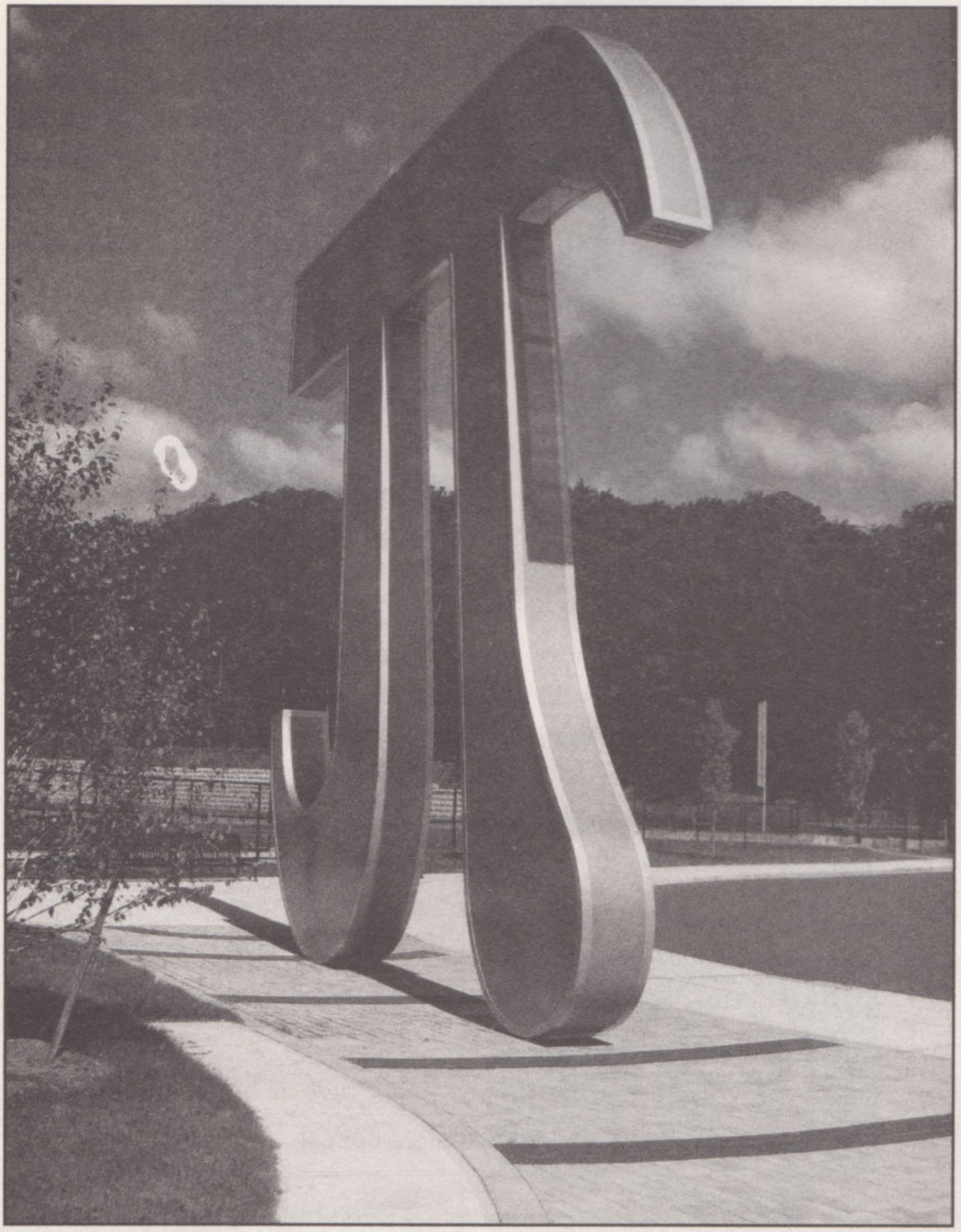
The true work of art based on π is in Toronto and is the brainchild of Canadian artist Arlene Stamp: the lobby of the Downsview subway station is a giant mosaic, made from rectangles, all of the same width, overlapping each other. The distance at which they do it is not capricious, though without knowing in advance it is hard to tell. Each rectangle covers the next, leaving only part of the total visible. If the total is width of each is 1, the part left visible is proportional to one decimal place of π. The mosaic starts at .1, the first decimal place of 3.1415926535…, and follows the order of the digits of our miraculous number.

As π is random (or seems that way), no one can guess the system used to place each rectangle. But, as noted in his day by mathematician Ivars Peterson, there is order amid the apparent randomness.



The coloured rectangles are all the same, but they are placed in such a manner that their visible surface is proportional to the corresponding digit of π. Each tile acts as a screen, hiding the previous one.

At the door of the Henry Abbott Technical School, located in Danbury, Connecticut, there is a statue of π almost 20 m in height, erected by sculptor Barbara Grygutis. It is illuminated at night and, no doubt, is a constant reminder to the students of the power for π.



Sculpture of π by Barbara Grygutis in Connecticut.

In another American city, Seattle, a statue of π was temporarily situated at the harbour front, on the steps next to the Museum of Art.

On the entrance of the Technical University in Berlin there is also a mosaic of the constant.

There is a nother work of purely mathematical art:

$$e^{\pi i} + 1 = 0,$$

which combines in a single expression the most notable five constants (e, π, i, 1 and 0). Considered the most beautiful formula of all science, the first to reveal it as well as prove it was the Swiss mathematician Leonhard Euler.

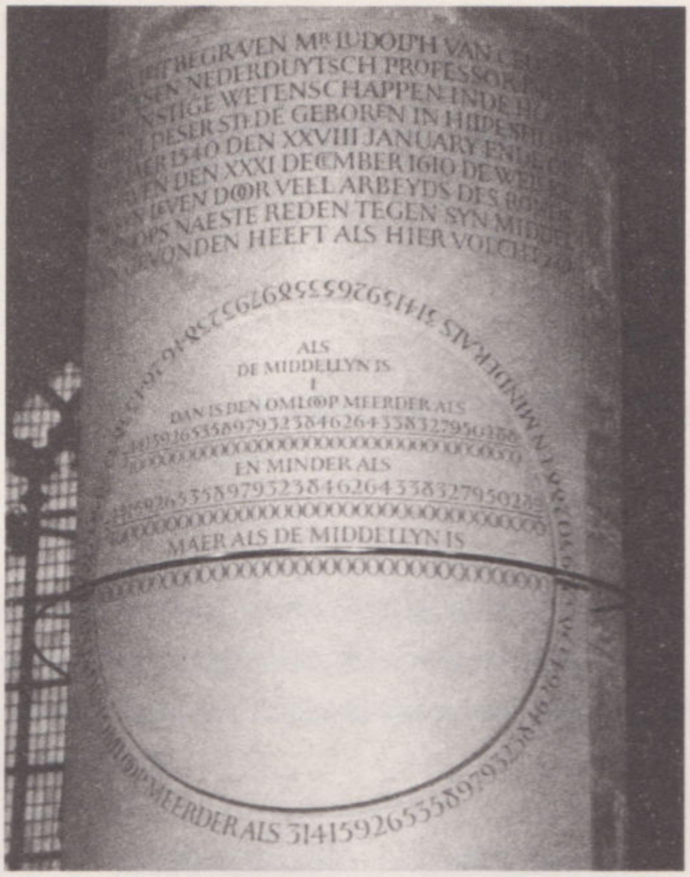
Perhaps Ludolph van Ceulen (1540-1610) would not share that view. After

LEONHARD EULER (1707-1783)

The Swiss Euler was one of the most eminent mathematicians in history. He was so intelligent that after he became blind, his phenomenal memory and mental calculation allowed him to continue working until his death twenty years later. He died in Russia, where he had been brought by the Empress Catherine II the Great, seduced by her fame and also the reports of a Swiss colleague Daniel Bernoulli. He lived there on and off from 1737. Euler's complete works (only the scientific ones), are compiled into 80 volumes. Euler was the one who definitively baptised pi with the Greek symbol π, and also introduced other notations, such as $f(x)$ for the functions of the variable x, i for the imaginary number unit, the constant e and the sum symbol Σ.

His other contributions are almost countless and have made an indelible imprint in fields such as calculus, functions, number theory, topology, graph theory, physics and astronomy; there is even an asteroid named after him. He also discovered many series related to the number π.

A stamp issued in the Soviet Union in 1957 to commemorate the 250th birthday of Euler.


BEGRAVEN MR LUDOLPH VAN
NEDERDUYTSCH PROFESSOR
STIGE WETENSCHAPPEN INDE
DESER STEDE GEBOREN IN HILDESH
1540 DEN XXVIII JANUARY
DEN XXXI DECEMBER 1610 DE
LEVEN DOOR VEEL ARBEYDS DES
NAESTE REDEN TEGEN SYN MIDD
VONDEN HEEFT ALS HIER VOLCHT
ALS
DE MIDDELLYN IS
1
DAN IS DEN OMLOOP MEERDER ALS
EN MINDER ALS
MAER ALS DE MIDDELLYN IS
MEERDER ALS 314159265358979

The tomb of Ludolph van Ceulen in Leiden, with the 35 digits of pi that he calculated.

working hard with circles and polygons with 2^{62} sides, the German mathematician came to an approximation of π, first with 20 and then with 35 decimal places. Such was his fervour and enthusiasm that he ordered that the number be carved on his tombstone for the edification of future generations.

Chapter 6

A second glance at infinity

Computers are useless. They can only give you answers.
Pablo Picasso

Let us now turn to the infinite nature of π, an infinity that may seem very familiar and probably requires less imagination than that used by Georg Cantor.

The Nobel Physics laureate Richard Feynman (1918-1988) found a curious string of straight nines in π:

3.1415926535897 9323846264338 3279502884197 1693993751058
2097494459230 7816406286208 9986280348253 4211706798214
8086513282306 6470938446095 5058223172535 9408128481117
4502841027019 3852110555964 4622948954930 3819644288109
7566593344612 8475648233786 7831652712019 0914564856692
3460348610454 3266482133936 0726024914127 3724587006606
3155881748815 2092096282925 4091715364367 8925903600113
3053054882046 6521384146951 9415116094330 5727036575959
1953092186117 3819326117931 0511854807446 2379962749567
3518857527248 9122793818301 1949129833673 3624406566430
8602139494639 5224737190702 1798609437027 7053921717629
3176752384674 8184676694051 3200056812714 5263560827785
7713427577896 0917363717872 1468440901224 9534301465495
8537105079227 9689258923542 0199561121290 2196086403441
8159813629774 7713099605187 072113**4999999**…

This occurs at point 762 and is called, appropriately enough, 'Feynman point'. The randomly likelihood of six nines in a row coming up is very low, only 0.08%. Does the Feynman point have any significance? It is just another of π's mysteries.

Another case of rare privilege is the sequence 0123456789, which was found at position 17,387,594,880. This time it was not Feynman who found it, but a computer program.

Now we will construct a magic square, that is, a square where the rows, columns and diagonals all add up to the same number, in this case, 65:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

This magic square was designed by the American T. E. Lobeck.

Now we move to the decimals of π. For each number in the square, call it *n*, we take the digit that we find in this position (the *n*th digit) of the decimal expression of π. If we look at the first number in the square, 17, we go to decimal digit 17, which happens to be a 2. We put it in the place of the 17, and so on. Thus we get another square on whose margins we will write down the sums of the rows and columns:

2	4	3	6	9	(24)
6	5	2	7	3	(23)
1	9	9	4	2	(25)
3	8	8	6	4	(29)
5	3	3	1	5	(17)
(17)	(29)	(25)	(24)	(23)	

The surprise comes when we see that the sum of each column is also the sum of each row. It looks like sleight of hand, but in mathematics there is no such thing as magic. Why does this happen? That is also unknown. We just know so little of π and infinity…

Monkeys, keyboards and libraries

Let's take a walk down a path through what we do not know – and we might never know – about π. Besides elevating us to the peak of human thought and to the very border of the unknown, this tour allows us to move into the realms of anecdote and legend.

Darwin and Darwinism have a special place in Anglo-Saxon tradition. By publishing his book *The Origin of the Species,* Charles Darwin offended many 'right-thinking spirits' by suggesting that human evolution did not follow a different path from other animals. The phrase "man descended from monkeys" is a gross simplification of the Darwinian theory, but has remained a biological paradigm. Any reference to monkeys suggests scientific controversy. Moreover, the mention of the keyboard of a typewriter in Victorian times resonated with modern technology. Join the two, monkeys and keyboards, and we get a potentially explosive cocktail. The story of monkeys and keyboards is a classic thought experiment with its roots entwining both the old Darwinian ideas with modern ones.

Suppose each member of a family of monkeys is presented with a keyboard. Ignoring the reality, which would be far from encouraging – experiments have been done with macaque monkeys which produced large amounts of typed pages full of the letters, which for some mysterious reason is impressive to them. Suppose, therefore, that the monkeys behave as would be expected of responsible animals, and type in more or less random symbols on blank sheets of paper, and that the process goes on and on.

The so-called 'monkey keyboard theorem' states that if the typing time is infinite, the monkeys will almost certainly write an intelligible copy of a previously written text. That is, if there is no time limit, the group of monkeys typing at the keyboard would end up, for example, typing *Romeo and Juliet* exactly, letter by letter – or even the words in this book.

In probability theory 'almost certainly' means 'borderline' or 'with probability tending to one'. It has an unambiguous meaning, all very mathematical. Now, this does not preclude that at any given moment and despite having waited an unimaginably long time, we look at what the monkeys have typed and are disappointed because they have not yet written anything intelligible. The only thing that makes the theory work is that the probability tends to one (something close to one hundred percent) if time marches on infinitely.

A much more literary version, related to this theorem, is in a story by Jorge Luis Borges, Argentinian writer, teacher and explainer of the infinite to whom we

An army of monkeys could type, if infinite time were available, any finite text imaginable.

owe so many visionary approaches to the universe. In *The Library of Babel*, a short story from the book *Ficciones* (1944), Borges describes a library that contains all possible books. It is sufficient to limit the size of the book to N letters or signs, with N large enough to fit into encyclopaedic tomes. According to the text of Borges, in this library are all permutations of the basic writing signs (characters, punctuation and white-space word separators) printed in book form. To be precise, here are some necessary calculations; the library imagined by Borges would contain about $25^{1132000} \approx 1{,}956 \cdot 10^{1834097}$ books at least.

We will not present here a rigorous and detailed proof of the theorem of keyboard tapping monkeys; it would be too long and require a somewhat boring exposition of the concepts of probability theory, but, broadly speaking, one can already guess that the content of the theory is true. Suppose that the keyboard had 60 keys. A simple sequence of signs, like "To be or not to be," from *Hamlet*, has 18 characters, and the probability of not typing it in n attempts is represented by

$$\left(1-\frac{1}{60^{18}}\right)^{n},$$

the limit for $n \to \infty$ of the entire expression is zero. If, for simplicity, we assume that instead of one monkey there are k monkeys, the limit remains unchanged, but intuitively you just arrive earlier; their convergence is accelerated.

More sophisticated considerations lead to the expected conclusion: There is no mathematical doubt that, with infinite time, the monkeys end up reproducing *Hamlet*. Any book imaginable is basically a sequence of letters, some variations with repetition of signs, of finite size. Infinite monkeys would write any book if the time to do so were infinite.

But it is also true that from the standpoint of daily life it is very unlikely that this situation could occur within an accessible time frame. What good does it do to know that monkeys can reproduce *War and Peace* if the time they need to achieve it is even older than the age of the universe? To write something meaningful, not necessarily Shakespeare, but a simple sentence, the monkeys would surely require an unthinkable period of time.

In fact, an expert on physical thermodynamics tell us that the reproduction of *Hamlet*, for example, is mathematically probable, but physically impossible. In fact, the universe is finite in particles and finite in time. Although there is a googol (a term coined by a nephew of mathematician Edward Kasner, and equivalent to 10^{100}) of particles and the Big Bang occurred at least 10 billion years ago, even if we replace the particles with monkeys and fill the universe with technological typing simians, they have a negligible probability of typing *Hamlet* in the history of the universe so far.

Borges's story in *The Library of Babel* is much more poetic and worth

The Argentine writer Jorge Luis Borges described a vast library of which in each cubicle there are 410 books of the same length.

more than our mere words; here is a descriptive paragraph about this wonderful library:

> "Everything will be in their blind volumes. Everything: the minutely detailed history of the future, *The Egyptians* of Aeschylus, the exact number of times the water of the Ganges have reflected the flight of a falcon, the secret and true name of Rome, the encyclopedia that had Novalis built, my dreams and day dreams at dawn of 14 August 1934, the demonstration of Pierre Fermat's theorem, the unwritten chapters of Edwin Drood, these same chapters translated into the language spoken by Garamantes, the paradoxes that Berkeley came up with about time and did not publish, the iron books of Urizen, the premature epiphanies of Stephen Dedalus before a thousand-year cycle having nothing to say, the Gnostic gospel of Basilides, the song that the sirens sang, the faithful catalogue of the Library, the demonstration of the fallacy of this catalogue. Everything, but for every sensible line or proper news, there will be millions of senseless cacophony, verbal jumbles and inconsistencies. Everything but the generations of men can do without the rapid shelves, shelves that obliterate the day and in which chaotic lives, have granted a tolerable page."
>
> (Jorge Luis Borges, *The Library of Babel.*)

The infinite digits of π

What we have recounted is more or less entertaining but is not as far from π and its decimal development as it may seem at first glance. If, instead of feeding our beloved monkeys with a complex keyboard, we gave them a keypad, we would obtain more or less random sequences of not letters, but numbers. A string of numbers. And what is π if not a string of numbers?

Borges's library can be replaced with a library not of finite literary works (most of them would be collections of meaningless gibberish) we would get books whose content would be finite numerical sequences, with a beginning and end.

Now, there is a basic difference between these fictional monkeys or the pen of Borges and the decimal sequence of π. The difference is that it deals with finite fictional entities, and π is not one of these. It possesses an infinite number of decimals; there is an countable infinity of digits in π.

There is no typing monkey that could handle π, nor any library, as large as it may be, that could house π and its digits. We have looked over the balcony of the infinite, but the development of π watches us impassively from there. Unavailable.

The indemonstrable normalcy of π

Is π normal? Paradoxically, the yet unanswered question that many mathematicians pose is whether π is a normal number An irrational number is said to be normal in base 10 when the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 appear in its decimal expansion at the same frequency, and the same being true of combinations of two numbers, 00 to 99, three numbers 000 to 999, and so on.

When a number is normal, written in all number bases it is said to be absolutely normal.

When Kanada calculated a million million digits of π he took a count of the amount of times each digit had appeared:

Decimal digit	Number of times the digit appears
0	99,999,485,134
1	99,999,945,664
2	100,000,480,057
3	99,999,787,805
4	100,000,357,857
5	99,999,671,008
6	99,999,807,503
7	99,999,818,723
8	100,000,791,469
9	99,999,854,780
Total	1,000,000,000,000

The distribution of the digits performed by Kanada does not show up particular irregularities that indicate π deviates from normal, although a sample of one billion items may seem insignificant.

Now, one thing is to guess and another to demonstrate, and the truth is that, although suspected, it has not been proven that π is normal.

In fact, it has not been proven that π, or *e*, or $\sqrt{2}$, or *log* 2, or even the golden ratio (Φ) or that any of the more current constants are normal. Interestingly, it has never been proven that any number is actually normal apart from the ones built by people expressly for that purpose. In 1917 the Polish mathematician Waclaw Sierpinski (1882-1969) showed the first normal number. It was absolutely normal.

Chaitin's constant,

$$\Omega = 0.00787499699...,$$

measures the probability that a randomly selected program stops a Turing machine. Its definition is frankly complicated, as it requires knowledge of how sums work, the bits of a program, the universal Turing machine, and other things. But, in reality, Ω is also a normal number, even though the above definition may not seem so.

Normal numbers are not that rare, since there is an uncountable infinity of them. The infinity of normal numbers is comparable to that of all real numbers. Almost all numbers are normal, what happens is they are very difficult for mathematicians to find. The statement that everything algebraic and irrational is normal belongs to the realm of speculation.

The inadequate randomness of π

The randomness of π seems clear, but it only seems that way. The Chudnovsky brothers, specialists in π, submitted its digits to all of the randomness tests imaginable, and π always emerged triumphant. Random numbers, to understand them, are those with digits that appear to be the fruit of randomness, have been tried and defined in many ways and for a long time. In the end Andrei Kolmogorov's definition seems pretty fitting. This emphasises complexity over chance. For Kolmogorov, a number is all the more complex the longer the minimum computer program that is used to describe it. Clearly, if to describe a number, the algorithm or minimum process that we apply is as long as the number, it must be very complex (or very random). If to calculate *N* we have to give instructions that are the value of *N*, then

ANDREI KOLMOGOROV (1903-1987)

Kolmogorov was born in the Russian city of Tambov; his mother died in childbirth and his father was deported for revolutionary activities, so he was raised by his aunts. In the 1930s he had already made a name in the international mathematical scene by publishing *Foundations of Probability Theory*, setting forth the material in an axiomatic but very modern way. His fame increased after he solved, along with one of his students, V.I. Arnold (b. 1937), problem #13 of Hibbert's famous list. (In 1900 David Hilbert, at the time the world's greatest mathematician, published a list with 23 major problems yet to be solved). Stochastic phenomena and Markov chains are among Kolmogorov's most studied mathematical subjects. His most innovative and difficult contribution was the theory of complexity, or randomness, which are two opposite sides of the same coin. His last years, having become a respected guru of Russian mathematics, were devoted to these concepts and to applied mathematics.

we should just write N right away and admit that we are dealing with a very complex, random number.

With π this doesn't exactly occur, because there are algorithms that calculate its digits (its individual pre-known digits!) and they are finite and relatively short; such that π should not be random in everything. For example, there is a 158-character computer program that can calculate 2,400 digit of π. We could say, in a simple way, that π may be random, but less so.

The digits of π appear one after the other in a manner that appears random. No pattern or configuration has been found that allows us to predict what will appear in a certain position in the number. Yes, there are procedures to calculate any figure, applying a BBP formula and the like, although this does not establish any pattern or anything like it. It is assumed that π must be 'weakly random' but no one has yet demonstrated it.

If every sequence of digits of π were random, the constant would be a normal number. But the opposite case is not generally true: a number can be normal and clearly not be random. The so-called Champernowne number, which will be discussed later, is normal and is not random, since there is a simple method that allows it to be constructed.

The inaccessible universality of π

Let's stick to base 10 for simplicity. A universe-number is a decimal that contains any sequence of numbers imaginable. If, by some procedure of coding, we turned the digits into letters, we could make π talk and find, in exploring its decimal expression, *Hamlet*, *Hello* magazine, this book or any of those conceived by Borges. All this along with the theses and rebuttals of the thesis, treaties and copies of treaties that differ only by one letter, incomprehensible gibberish and immense extensions of repeated letters. Everything would be somewhere, waiting passively for all eternity to be or not to be read.

Let's return for a few moments to Carl Sagan and his novel *Contact*. It is assumed that the aliens board Dr. Arroway's ship are in search of a message, which should be of divine origin because it is coded at the birth of the universe in the decimals of π (written in base 11, but we will not go into that detail). As we move deeper into the expansion of π, many millions of characters after the decimal point, there is a string of figures which, properly disposed in a plane, only consists of ones and zeros, and these shape or draw a square of ones with a circle of zeros inside. With the appearance of the square does it really show something as controversial as the existence of God?

If you add the first 20 digits of π, it equals 100. Going further, to digit 144, is the number of the Beast, the apocalyptic 666, which appears in the Bible. But we have no right to grant significance to those coincidences, no matter how divine they seem, because the truth is that it depends on the numbering system, and that's a very human system.

As human as David Gawen Champernowne (1912–2000), who devised a normal number in base 10. He found the constant that carries his name at the age of 21, before he had even finished his degree. It is an example of a transcendental number that is also a normal number and a universe number. It is defined in a manner so simple that it doesn't seem real, just type, one after the other, all the numbers in their natural order:

$$C_{10} = 0.1234567891011121314151617181920212223\ldots$$

There is no doubt that we find ourselves in the presence of a universe number, as any sequence N of digits imaginable would be found in the development of Champernowne's number. We have called it C_{10}, because this is its most common abbreviation in mathematics: the 10 indicates that it is written in base 10. Besides, there are infinite variations of C_{10}, all of them just as much a universe numbers as it is. We leave it to the imagination of the reader, so you may search for them as an exercise. We should add that the so-called Copeland-Erdös number is constructed like the Champernowne number, although limited to prime numbers, and is also normal (in base 10):

0 2 3 5 7 11 13 17 19 23 29 31...

From this point of view, the development of π described by Sagan looses strength. It is not so rare to find numbers that contain a circle of zeros and ones. In fact, they are endless, but do not reflect the circumference to diameter ratio, something that π does.

And is π a universe number? No own knows. We only know that all universe-numbers are normal, but that does not help much.

What can and can not be proven

The Czech logician and mathematician Kurt Gödel (1906-1978) proved something very puzzling, something that set limits for the first time to human knowledge. Imagine a logical system, with its theorems and axioms, which contains basic arithmetic in its propositions. For example, conventional maths, is it even imaginable that it is contradictory at some point? What nonsense! most of us would say. Is it imaginable that is incomplete? Can we entertain unprovable or disproved statements using the same tools of the system? Most people would also reject that. How could a field of thought, which contains the rules for basic arithmetic be incomplete? Every theorem is true or false, and maybe it takes us a long time to figure it out, but one day it will become clear... A good example is Fermat's Last Theorem: it took a lot of waiting, but finally the proof came.

Well, Gödel proved that such a system is either inconsistent or incomplete and cannot be both consistent and complete. If it is complete and we can prove or disprove everything, then at some point it is contradictory, and if it is flawless, without a shadow of contradiction, then it is incomplete. There will always be some proposition that is impossible to prove or disprove.

THE CONTINUUM HYPOTHESIS

George Cantor passed a good part of his life trying to prove a hypothesis, which can been set out thus: Let's call *A* a countable set, whose cardinality will be $\aleph_0$. We define $\aleph_1$ as the cardinality of $\wp(A)$, with $\wp(A)$ as the set of the parts of *A*:

$$\#\wp(A) = \aleph_1.$$

Now let's call *c* the non-countable cardinality of the real numbers, named the real continuum. Cantor arrived at the inequality:

$$\aleph_0 < c \leq \aleph_1.$$

And he firmly believed that between $\aleph_0$ and $\aleph_1$ no cardinality could exist, for which $c = \aleph_1$. That is the continuum hypothesis.

In 1963, American mathematician Paul Cohen (1934-2007) proved that this hypothesis is undecidable, and, therefore, a choice can be made between the truth or its falsity without changing anything in the field of conventional mathematics.

It is a curious situation that Gödel has put us in. Bertrand Russell jokingly defined mathematics as an activity in which no one knew what they wanted to prove or if anything proven was true. Gödel seems to have nailed that up even more tightly. We do not even know whether we will be able to prove something someday. Gödel's theorem is no fantasy, as some of those unprovable statements have already been found, including the so-called continuum hypothesis.

Surely no unprovable propositions are to be found among the so-called normals. If the consequence of something being unprovable in any way affects other issues of normal mathematics (Fermat's theorem is a good example) it is unlikely that we are facing a Gödelian proposition.

Let us examine the last questions raised by π. Do they have answers? Not now, no. Will they have in the future? Perhaps.

Not that we claim the unknowns about π are unprovable. Many think that they would only be so if their affirmation or denial had no impact on 'conventional' mathematics.

KURT GÖDEL (1906-1978)

Kurt Gödel and Albert Einstein at the Institute for Advanced Study in Princeton, New Jersey.

This Czech-born mathematician later became an American and specialised in logic. In Europe he worked with the so-called Vienna Circle and emigrated to America with the arrival of Nazism. The publication *On Formally Undecidable Propositions of Principia Mathematica and Related Systems* made him famous. Reading it, however, is so technical that a book by Ernest Nagel, a popular work (*Gödel's Proof*) is better known than the original article. In it Gödel explained his incompleteness theorems that show that a logical system large enough to include among its axioms of elementary arithmetic can not be both consistent and complete. It proved also that within such a system the consistency of the axioms cannot be proved.

Both findings are shocking even today and put a limit on the mathematical field in a manner similar to the way the Heisenberg Uncertainty Principle limited physics.

This result, and those of others that followed him, made Gödel an almost mythical figure among scientists worldwide. Even a book by Douglas R. Hofstadter, *Gödel, Escher, Bach: An Eternal, Graceful Loop*, has become a bestseller.

Throughout his life, Gödel developed a series of depressions and suffered paranoia that culminated in his rather dramatic death. He refused to eat anything that had not been tried first by his wife, so when she fell ill and was admitted to hospital, Gödel refused food and died of hunger.

Chapter 7

The First Ten Thousand Digits of π

> *The non-calculated decimals of π sleep in a mysterious abstract realm, where they enjoy a weak reality. They do not become real until they are calculated, and even then their reality is a simple question of degree.*
>
> William James

$\pi = 3.$ number of digits

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The Secrets of π

Why the circle can't be squared

You may remember it as 22 ÷ 7, or 3.14159.... but what is it? Better known as π, this number describes the relationship between the diameter of a circle and its circumference. It is a number that has fascinated mathematicians since the time of the Ancient Greeks, and has aroused more controversy and debate than any other number.